

10/26/09

Math 250A

Pg. 1

Existence + Uniqueness Theorem

(first order ODEs)

$$\frac{dy}{dx} = g(x, y), \quad y(x_0) = y_0$$

Implementation (see L.L. pg. 57)

↓
 solution exists?
 if so, is it 'unique'?

1 confirm that $g(x, y)$ is continuous in the vicinity of (x_0, y_0)

2 Compute $\frac{\partial g}{\partial y}$ and is cont. . . . "

3 If #1 and #2, then a unique solution exists. If not, we are not sure (i.e., there may or ^{may} not be a solution, that is or is not unique)

Partial Derivatives $g(x, y) \rightarrow$ function of more than one variable!

Rule: to differentiate, keep one variable 'const.'

ex $g(x, y) = (xy)^{\frac{1}{2}} \rightarrow \frac{\partial g}{\partial y} = \frac{1}{2} \sqrt{\frac{x}{y}}$

$g = x^{\frac{1}{2}} y^{\frac{1}{2}} \rightarrow \frac{\partial}{\partial y} g = x^{\frac{1}{2}} \frac{\partial}{\partial y} y^{\frac{1}{2}} = x^{\frac{1}{2}} \cdot \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2} \sqrt{\frac{x}{y}}$

$$\frac{\partial g}{\partial x} = \frac{1}{2} \sqrt{\frac{y}{x}}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial g}{\partial y} \right) = \frac{\partial}{\partial x} \frac{1}{2} \sqrt{\frac{x}{y}} = \frac{1}{2} \frac{\partial}{\partial x} x^{\frac{1}{2}} y^{-\frac{1}{2}}$$

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$$= \frac{1}{2} y^{-\frac{1}{2}} \frac{\partial}{\partial x} x^{\frac{1}{2}} = \frac{1}{4} y^{-\frac{1}{2}} x^{-\frac{1}{2}} = \frac{1}{4} \sqrt{\frac{1}{xy}}$$

$$\frac{\partial}{\partial y} \left(\frac{\partial g}{\partial x} \right) = \frac{1}{4} \sqrt{\frac{1}{xy}}$$

ex $\phi(a, b) = ab^2 + 3a^2e^b \rightarrow \text{find } \frac{\partial}{\partial b} \left(\frac{\partial \phi}{\partial a} \right)$

$$\begin{aligned} \frac{\partial}{\partial a} \phi &= \frac{\partial}{\partial a} (ab^2 + 3a^2e^b) &= 2b + 6ae^b \\ &= b^2 + 6ae^b \end{aligned}$$

$$\frac{\partial}{\partial b} \frac{\partial \phi}{\partial a} = \frac{\partial}{\partial b} (b^2 + 6ae^b) = 2b + 6ae^b \therefore$$

Exist. + Uniq. Examples

ex $\frac{dy}{dx} = y^3 = g, \quad y(0) = 0$

- y^3 is cont. in vicinity of $y(0) = 0$
(actually cont. everywhere!)

- $\frac{\partial g}{\partial y} = 3y^2 \rightarrow$ cont. everywhere

$\Rightarrow$ there is a unique solution to $\frac{dy}{dx} = y^3$ that passes through the point $y(0) = 0$

$$\boxed{\text{ex}} \quad \frac{dy}{dx} = (xy)^{\frac{1}{2}} = g, \quad y(1) = 1$$

- $g(x,y)$ is cont. when $\begin{cases} x=y=0 \\ x,y \geq 0 \\ x,y \leq 0 \end{cases}$

$$- \frac{\partial g}{\partial y} = \frac{1}{2} \sqrt{\frac{x}{y}} \rightarrow \text{cont. when } \begin{cases} x,y > 0 \\ x,y < 0 \end{cases}$$

$\rightarrow$ cont. when $\begin{cases} x \geq 0 \text{ AND } y > 0 \\ x \leq 0 \text{ AND } y < 0 \end{cases}$

$\rightarrow$ a unique solution exists for $y(1) = 1$

$y(-2) = +1 \rightarrow$ not sure if a (unique) solution exists

$\boxed{\text{ex}}$ True or False: There are at least three different solutions to $\frac{dy}{dx} = y^{\frac{1}{3}}$ passing through $(0,0)$