

250 Lecture

10/21/0

Goal: introduce/motivate the Existence-Uniqueness Theorem (see LL 2.4)

Theorem If $g(x, y)$ and $\frac{\partial g}{\partial y}$ are defined and continuous in a finite rectangular region containing the point (x_0, y_0) in its interior, then the ODE
$$y' = g(x, y)$$

has a unique solution passing through the point $y(x_0) = y_0$. This solution is defined for all x for which the solution remains inside the rectangle.

Rephrased assume $\frac{dy}{dx} = g(x, y)$

Existence: if $g(x, y)$ is a continuous function of x and y on the rectangle

$$R = \{ (x, y), |x - x_0| \leq a, |y - y_0| \leq b \}$$

where $a > 0$ and $b > 0$, then there exists a continuously differentiable solution y of the above ODE on $|x - x_0| \leq \alpha$ for which $y(x_0) = y_0$ where

$$\alpha = \min\left(a, \frac{b}{M}\right), \quad M = \max_{x, y \in R} |g(x, y)|$$

→ this is called the Cauchy-Peano theorem

Uniqueness: If g is a continuous function of x and if there exists a constant $K > 0$ such that for all x , y_1 and y_2 are in R , we have

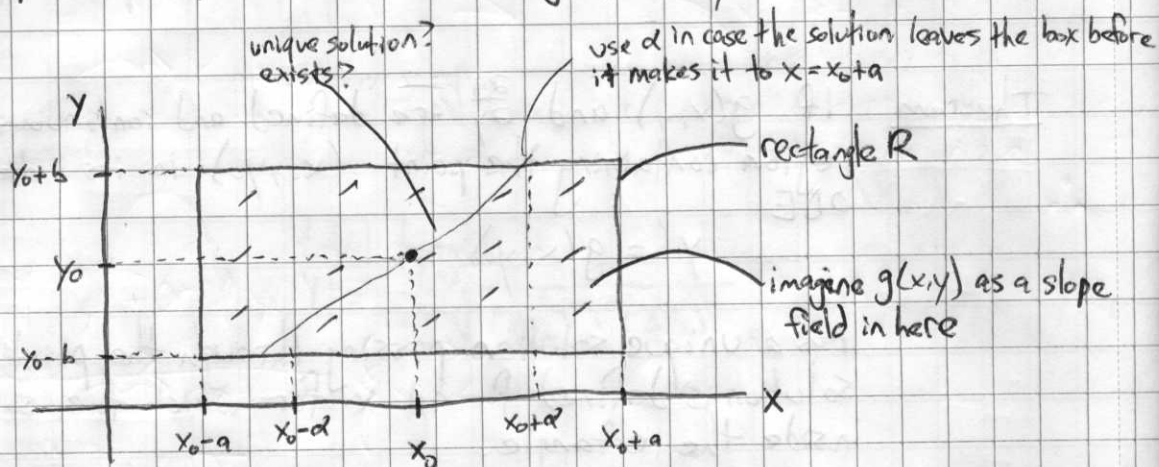
$$|g(x, y_1) - g(x, y_2)| \leq K |y_1 - y_2|$$

then there exists a unique solution y to the ODE on $|x - x_0| \leq \alpha$ such that $y(x_0) = y_0$. [R and α are same as defined in previous theorem]

→ this is called the Picard-Lindelöf theorem

10/21/09

→ let's interpret the previous theorems graphically:



→ in simplest terms, we are asking if $g(x,y)$ is 'well-behaved' inside our (arbitrarily) defined box

- if so (i.e. it is continuous and differentiable), things are straight-forward in that we have a solution and it is unique
- if not, we need to look more carefully at what is going on, as we may or may not have a solution, or at least one that is unique

□ practically, a unique solution means that if two solutions pass through the point (x_0, y_0) , those two solutions are identical

CAVEAT: $g(x,y)$ implies a function of more than one variable!!

→ might need compute 'partial derivatives' (topic of focus in vector calc. and PDEs)

$$\frac{\partial g}{\partial y} = \frac{d}{dy} g(x,y) \text{ where } x \text{ is considered const.}$$

[ex.] $z = (xy)^{1/2}$, then $\frac{\partial z}{\partial x} = \frac{1}{2} y^{1/2} \cdot x^{-1/2}$

$$\frac{\partial z}{\partial y} = \frac{1}{2} x^{1/2} \cdot y^{-1/2}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{1}{4} x^{-1/2} y^{-1/2} = \frac{1}{4\sqrt{xy}} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right)$$