

10/20/09

TUESDAY PROBLEMS

GROUP 1

Consider the ODE: $\frac{dy}{dt} = y(y-2)(y+1)$

- classify the ODE (nonlinear, autonomous)
- determine qualitative behavior of solutions (see 10/18 notes)
- determine explicit general solution $\rightarrow$ consistent w/ dfield?

SOLUTION

- separate variables and use PFD:

$$\int \frac{1}{y(y-2)(y+1)} dy = \int dt = t + C = \int \left[-\frac{1}{2y} + \frac{1}{6(y-2)} + \frac{1}{3(y+1)} \right] dy$$

$$= -\frac{1}{2} \ln|y| + \frac{1}{6} \ln|y-2| + \frac{1}{3} \ln|y+1| = t + C$$

$\Rightarrow$ not trivial (if even possible!) to express $y(t)$, but you can compute $t(y)$ and then plot (t, y) to show this expression is indeed consistent w/ what we get from dfield (I checked this in Matlab; choosing an arbitrary value of C just moves you along the t -axis as one might expect...)

GROUP 2

Use Riemann slices to find the Volume of a sphere of radius r .SOLUTION

$$\Delta V = \pi \alpha^2 \Delta x$$

$$= \pi(r^2 - x^2) \Delta x$$

$$\rightarrow V = \int_{-r}^r \pi(r^2 - x^2) dx$$

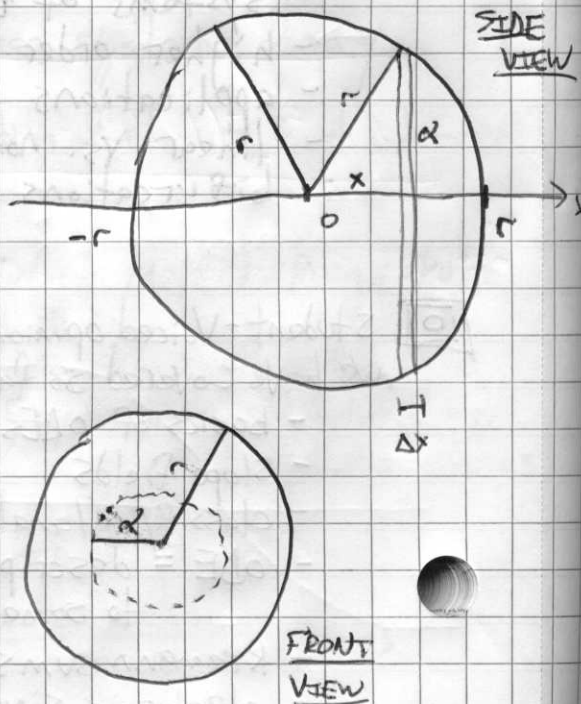
$$= \pi r^2 x \Big|_{-r}^r - \pi \cdot \frac{1}{3} x^3 \Big|_{-r}^r$$

$$= \pi \left[2r^3 - \frac{1}{3} \cdot 2r^3 \right] = \pi r^3 \left[2 - \frac{2}{3} \right]$$

$$= \frac{4}{3} \pi r^3$$

$$x^2 + \alpha^2 = r^2$$

$$\rightarrow \alpha = \sqrt{r^2 - x^2}$$



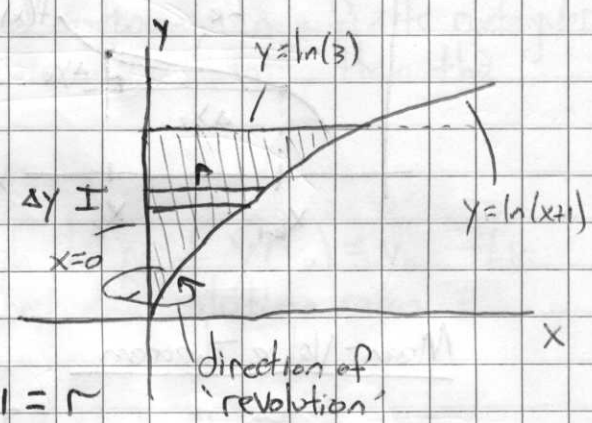
GROUP 3**'Solids of Revolution'**

Find the resulting volume of the solid obtained by rotating the region enclosed by the following curves about the y -axis

$$y = \ln(x+1), \quad y = \ln(3), \quad x=0$$

SOLUTION

- volume is a pointed/rounded cone, like the top half of a Hershey kiss
- use horizontal slices to make use of circular
- Volume of slice: $\Delta V = \pi r^2 \cdot \Delta y$



$$\rightarrow \text{what is } r? \quad y = \ln(x+1) \rightarrow x = e^y - 1 = r$$

$$\Delta V = \pi (e^y - 1)^2 \Delta y \Rightarrow \cancel{V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^{\ln(3)} \pi (e^y - 1)^2 \Delta y} \quad V = \lim_{\Delta y \rightarrow 0} \sum_{y=0}^{\ln(3)} \pi (e^y - 1)^2 \Delta y = \int_0^{\ln(3)} \pi (e^y - 1)^2 dy$$

$$\rightarrow V = \int_0^{\ln(3)} \pi [e^{2y} - 2e^y + 1] dy$$

$$= \pi \left[\frac{1}{2} e^{2y} \Big|_0^{\ln(3)} - 2e^y \Big|_0^{\ln(3)} + y \Big|_0^{\ln(3)} \right]$$

$$= \pi \left[\frac{1}{2} e^{2\ln 3} - \frac{1}{2} - 2e^{\ln 3} + 2 + \ln 3 \right]$$

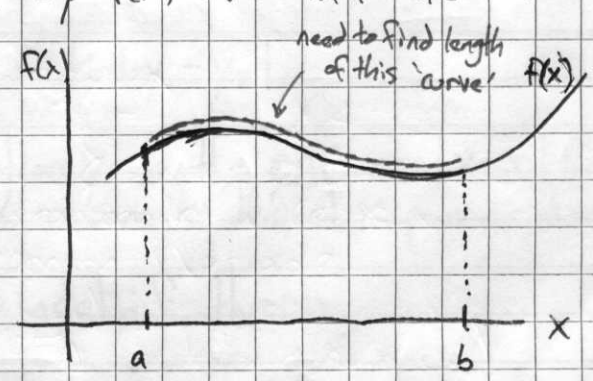
$$= \pi \left[\frac{9}{2} - \frac{1}{2} - 6 + 2 + \ln 3 \right] = \pi [4 - 4 + \ln 3] = \boxed{\pi \ln(3)}$$

GROUP 4**'Arc Length'**

Consider an arbitrary (but cont. and differentiable) function $f(x)$ on the interval $x \in [a, b]$. Find the length of the curve traced out by $f(x)$ over that interval.

SOLUTION

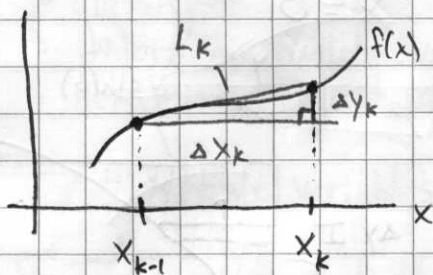
Strategy: just like using rectangles via Riemann sums to find areas, consider small line segments to get the curve length



10/20/09

Group 4 (cont.)

→ focus on a short segment going from x_{k-1} to x_k
where $a \leq x_{k-1} < x_k \leq b$



$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

total length:

$$L = \sum_{k=1}^N \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$\Delta x_k \equiv x_k - x_{k-1}$$

$$\Delta y_k \equiv y_k - y_{k-1}$$

Mean-Value Theorem

let $c_k \in [x_{k-1}, x_k] \rightarrow f'(c_k) \equiv \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$
(c_k is some value of x on the interval)

now $\Delta y_k = y_k - y_{k-1} = f(x_k) - f(x_{k-1})$, we have $\Delta y_k = f'(c_k) \Delta x_k$

$$\rightarrow L \approx \sum_{k=1}^N \sqrt{(\Delta x_k)^2 + (f'(c_k) \Delta x_k)^2} = \sum \sqrt{1 + (f'(c_k))^2} \Delta x_k$$

$$\Rightarrow L = \lim_{\Delta x \rightarrow 0} \sum L_k = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$\Rightarrow \boxed{L = \int_a^b [1 + (f'(x))^2]^{\frac{1}{2}} dx}$$