

- basic questions to ask given an ODE: (see LL 1.2, 1.3)
- monotonicity - when are solutions increasing/decreasing?
 - concavity - " " concave up/down?
 - symmetry - Are there symmetries in the solutions?
 - singularities - Are there places where the derivative blows up?
 - Uniqueness - Do solutions intersect?
 - Equilibria - where does $\frac{dy}{dx} = 0$?
(and what about their stability?)

ex $\frac{dy}{dx} = \frac{1}{x}$

- solutions increase for $x > 0$; decrease for $x < 0$

- second derivative:

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} \rightarrow \text{always negative; always concave down}$$

- $\frac{1}{x} \neq 0$ for all $x \rightarrow$ no equilibria

- check if things are symmetric about y-axis

$\rightarrow$ plug in $-x$ for x

$$\frac{dy}{d(-x)} = -\frac{dy}{dx} = \frac{1}{(-x)} = -\frac{1}{x}$$

about origin

$$\frac{d(-y)}{d(-x)} = \frac{dy}{dx} = \frac{1}{(-x)} = -\frac{1}{x}$$

- ~~as~~ as you approach $x=0$, slope of solution curve diverges

- Are solutions unique?

(mini) Proof: • Assume two distinct solutions to ODE, $y_1(x)$ and $y_2(x)$

• Assume both intersect @ a common point (x_0, y_0)

→ According to the ODE, $y_1' = \frac{1}{x}$ and $y_2' = \frac{1}{x}$
then $y_1' = y_2'$

Rewrite as $y_1' - y_2' = 0 = (y_1 - y_2)'$

$$\frac{d}{dx}(x^2 + x^3) = 2x + 3x^2$$

$$\rightarrow y_1(x) - y_2(x) = C$$

2nd assumption: $y_1(x_0) = y_0$ and $y_2(x_0) = y_0$

$$\rightarrow C = 0 \Rightarrow y_1(x) = y_2(x)$$

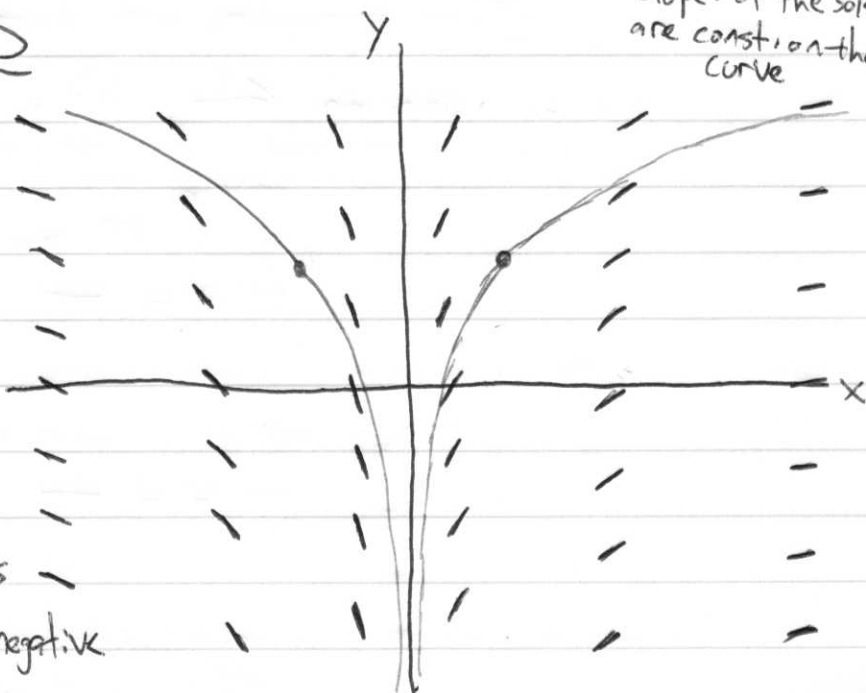
→ solutions are unique!!

SLOPE FIELD

$$\frac{dy}{dx} = \frac{1}{x}$$

- slopes independent of y
→ isoclines are vert. lines
- as $|x| \rightarrow 0$, slope gets big,
as $|x| \rightarrow \infty$, slope gets small
- slope undefined @ $x = 0$
- sign change about y -axis
- slopes positive for $x > 0$, negative for $x < 0$

isocline
curve where the
slopes of the sols.
are const. on that
curve



19.3

ex (cont.) $\frac{dy}{dx} = \frac{1}{x} \rightarrow y(x) = \int \frac{1}{x} dx$
 $= \ln|x| + C$ (general solution)

unique solution: need an initial condition (I.C.)

$y(x_0) = y_0$

$\rightarrow C = y_0 - \ln|x_0|$

$\rightarrow$ our analytic solution is consistent w/ everything we've deduced up to this point

∴

ex $\frac{dy}{dx} = \sqrt{1-y^2}$ w/ I.C. $y(0) = 0$

- solutions increase for $|y| < 1$
- no solutions for $|y| \geq 1$!!

• concavity? $\frac{\partial^2 y}{\partial x^2} = \frac{d}{dx} (1-y^2)^{\frac{1}{2}} = \frac{1}{2} (1-y^2)^{-\frac{1}{2}} \cdot -2y \cdot \frac{dy}{dx}$

$= - \frac{y}{(1-y^2)^{\frac{1}{2}}} \cdot (1-y^2)^{\frac{1}{2}} = -y$

$\rightarrow$ concave up solutions for $y < 0$, concave down for $y > 0$

• symmetric about the origin? $(x, y) \rightarrow (-x, -y)$
 $\frac{\partial(-y)}{\partial(-x)} = \frac{dy}{dx} = \sqrt{1-(-y)^2} = \sqrt{1-y^2}$

10/12/09 (cont.)

ex autonomous eqn. (ex. 2.10 in LL)

$$y' = \sqrt{1-y^2} \quad \text{with I.C. } y(0) = 0$$

• right off the bat, we notice we must have $|y| \leq 1$ to avoid getting a complex # (we are not presently equipped to handle such a case)

• for $|y| \leq 1$, solutions will be increasing since $y' > 0$

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{d}{dx} (1-y^2)^{\frac{1}{2}} = \frac{1}{2} (1-y^2)^{-\frac{1}{2}} \cdot (-2y) \cdot \frac{dy}{dx} \\ &= -y \end{aligned}$$

→ solutions will be concave up for $y > 0$ and concave down for $y < 0$

• symmetries? let $(x, y) \rightarrow (-x, -y)$

$$\text{then } \frac{dy(-x)}{d(-x)} = \frac{dy}{dx} = \sqrt{1-(-y)^2} = \sqrt{1-y^2}$$

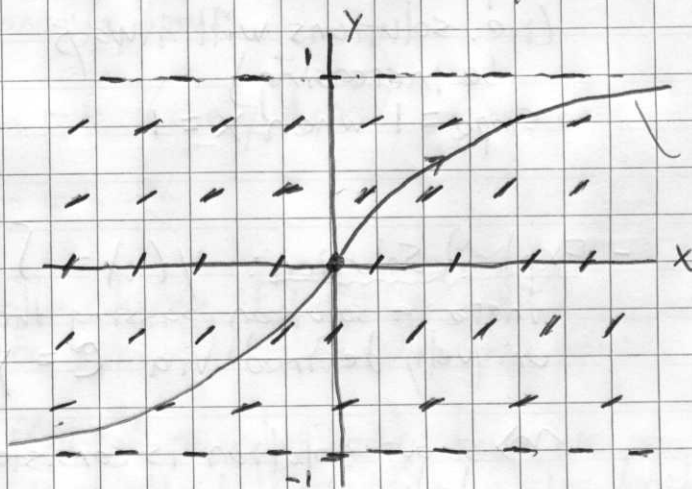
→ solution could be symmetric w/ respect to the origin

• as noted above, $|y| > 1$ presents a problem, so we wouldn't expect to find solutions there

• if $y = 1$ or $y = -1$, then $\frac{dy}{dx} = 0 \rightarrow$ two equilibria

• slope field:

- slope = 1 for $y = 0$
- slopes independent of x (i.e. horizontal isoclines)
- slope always positive
- slope increases as y moves away from $y = 0$



unique solution
passing thru
 $(x, y) = (0, 0)$

10/12/09

• Explicit Solution: separate variables + integrate:

$$\int \frac{1}{\sqrt{1-y^2}} dy = \int dx = x + C$$

$$\begin{aligned} \text{let } y &= \sin \theta \\ dy &= \cos \theta d\theta \end{aligned}$$

$$= \int \frac{1}{\sqrt{1-\sin^2 \theta}} \cdot \cos \theta d\theta = \int d\theta = \theta$$

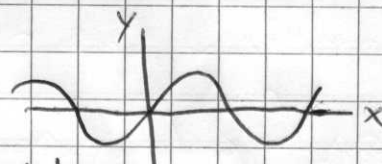
$$\rightarrow \theta = \arcsin y$$

$$\rightarrow \theta = \arcsin y = x + C$$

$$\rightarrow y(x) = \arcsin(\sin(x+C)) \quad (C=0 \text{ given our I.C.})$$

$$= \sin(x)$$

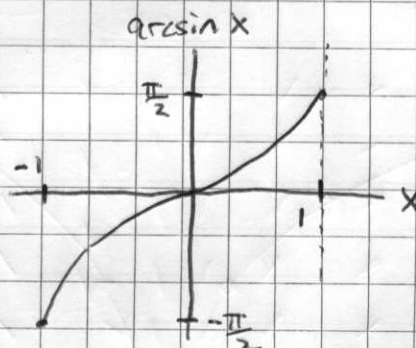
$\Rightarrow$ so our solutions should then look like
but this is totally inconsistent w/ our
slope field; thus $y(x) = \sin(x)$ is not our solution!



PROBLEM

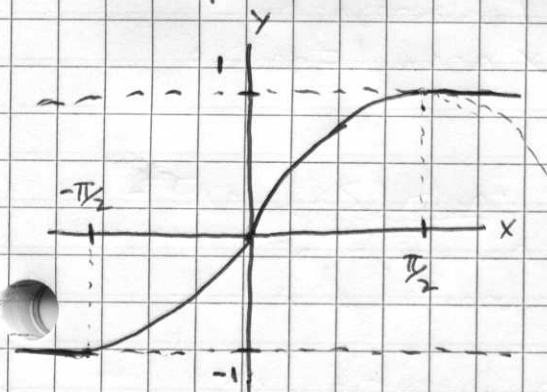
BE CAREFUL!

$\rightarrow$ remember what $\arcsin$ looks like
(see 10/6/09 notes); the range
of that function is $[-\frac{\pi}{2}, \frac{\pi}{2}]$,
thus it is only valid for
 $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$



$\rightarrow$ so our solution is really $y(x) = \sin(x)$ where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

• but surely x can be larger than $\frac{\pi}{2}$ (or less than $-\frac{\pi}{2}$), so what happens there? [draw a picture!]



$\rightarrow$ at $x = \pm \frac{\pi}{2}$, we get onto the equilibrium
solution tracks, thus

$$y(x) = \begin{cases} -1 & x < -\frac{\pi}{2} \\ \sin x & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x \end{cases}$$

$\rightarrow$ in this ODE, solutions can exist that
are not necessarily unique!!