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let $x = \sin \theta = \cos(\theta - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - \theta)$
 [since $\cos(-x) = \cos(x)$]

then $\theta = \arcsin x$

and $\frac{\pi}{2} - \theta = \arccos x = \frac{\pi}{2} - \arcsin(x)$

thus $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ OR $\arcsin(x) = -\arccos(x) + \frac{\pi}{2}$

3 Problems for students to work on in groups:

P1 Solve the ODE: $\frac{dy}{dx} = \frac{x^5 + x^4}{(x^2 + 9)^2}$

• separate, use long division to simplify:

$$x^4 + 18x^2 + 81 \overline{) x^5 + x^4} \quad \rightarrow \quad \frac{x^5 + x^4}{(x^2 + 9)^2} = 1 + x - \frac{18x^3 + 18x^2 + 81x + 81}{(x^2 + 9)^2}$$

$$\underline{-18x^3 - 18x^2 - 81x - 81}$$

• use Partial fractions on the last piece:

$$\frac{18x^3 + 18x^2 + 81x + 81}{(x^2 + 9)^2} = \frac{Ax + B}{x^2 + 9} + \frac{Cx + D}{(x^2 + 9)^2}$$

(multiply both sides by $x^2 + 9$ and set $x=0$): $\frac{81}{9} = B + \frac{D}{9}$

(multiply both sides by $(x^2 + 9)^2$ and set $x=1$): $198 = 10A + 10B + C + D$ (not too helpful?)

$$\begin{aligned} 18x^3 + 18x^2 + 81x + 81 &= (Ax + B)(x^2 + 9) + Cx + D \\ &= x^3(A) + x^2(B) + x(C + 9A) + (9B + D) \end{aligned}$$

$\rightarrow A = 18, B = 18, C = 81 - 9(18) = -81, D = 81 - 9(18) = -81$

$$\rightarrow \frac{18x^3 + 18x^2 + 81x + 81}{(x^2 + 9)^2} = \frac{18(x+1)}{x^2 + 9} - \frac{81(x+1)}{(x^2 + 9)^2}$$

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[P1] (cont.) thus $\frac{x^5+x^4}{(x^2+9)^2} = 1+x - \frac{18x^3+18x^2+81x+81}{(x^2+9)^2}$

$$= 1+x - \frac{18(1+x)}{x^2+9} + \frac{81(1+x)}{(x^2+9)^2}$$

I
easy to
integrate

II
you have the
pieces for this

III
trickier...

I $\int (1+x) dx = x + \frac{x^2}{2} + C$

II $\int \frac{1+x}{x^2+9} dx = \int \frac{1}{x^2+9} dx + \int \frac{x}{x^2+9} dx$ (let $u = x^2+9$)
 $\begin{aligned} \text{let } x &= 3 \tan \theta \\ dx &= 3 \frac{1}{\cos^2 \theta} d\theta \end{aligned}$
 $= \frac{1}{3} \int d\theta + \int \frac{1}{u} \cdot \frac{1}{2} du$
 $= \frac{1}{3} \theta + \frac{1}{2} \ln|u| + C$
 $= \frac{1}{3} \arctan\left(\frac{x}{3}\right) + \frac{1}{2} \ln|x^2+9| + C$

$\rightarrow \int -\frac{18(1+x)}{(x^2+9)^2} dx = -6 \arctan\left(\frac{x}{3}\right) - 9 \ln|x^2+9| + C$

III $\int \frac{1+x}{(x^2+9)^2} dx = \int \frac{x}{(x^2+9)^2} dx + \int \frac{1}{(x^2+9)^2} dx$
 solved via a substit. trickier; see [P3] on next pg.

(when the smoke clears) $= -\frac{1}{2} \frac{1}{x^2+9} + \frac{1}{18} \frac{x}{x^2+9} + \frac{1}{18} \cdot \frac{1}{3} \arctan\left(\frac{x}{3}\right) + C$

$\rightarrow \int \frac{81(1+x)}{(x^2+9)^2} dx = -\frac{81}{2} \frac{1}{x^2+9} + \frac{9}{2} \frac{x}{x^2+9} + \frac{3}{2} \arctan\left(\frac{x}{3}\right) + C$

$\Rightarrow$ putting it all together (and simplifying things a bit algebraically):

$y(x) = x + \frac{1}{2}x^2 - \frac{9}{2} \arctan\left(\frac{x}{3}\right) - 9 \ln|x^2+9| + \frac{9}{2} \frac{(x-9)}{x^2+9} + C$

[these two questions below lead into P1's solution]

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[P2] Solve the ODE: $\frac{dy}{dx} = \frac{1}{x^2+9}$

$$y(x) = \int \frac{1}{x^2+9} dx$$

[use trig. substit.: let $x = 3 \tan \theta$
 $dx = \frac{3}{\cos^2 \theta} d\theta$]

$$= \int \frac{1}{9 \tan^2 \theta + 9} \cdot \frac{3}{\cos^2 \theta} d\theta = \frac{1}{3} \int \frac{1}{(\frac{\sin^2 \theta}{\cos^2 \theta} + 1) \cos^2 \theta} d\theta$$

$$= \frac{1}{3} \int d\theta = \frac{1}{3} \theta + C = \boxed{\frac{1}{3} \arctan\left(\frac{x}{3}\right) + C}$$

$\therefore$

[P3] Solve the ODE $\frac{dy}{dx} = \frac{1}{(x^2+9)^2}$

$\rightarrow$ requires one to think outside the box - - - -

try using integration by parts on [P2]:

$$\int \frac{1}{x^2+9} dx$$

let $u = (x^2+9)^{-1}$
 $v' = 1$

$u' = -(x^2+9)^{-2} \cdot 2x$
 $v = x$

$$= \frac{x}{x^2+9} - \int \frac{-2x^2}{(x^2+9)^2} dx$$

$$= \frac{x}{x^2+9} + 2 \int \frac{x^2}{(x^2+9)^2} dx = \frac{x}{x^2+9} + 2 \int \frac{(x^2+9) - 9}{(x^2+9)^2} dx$$

$$= \frac{x}{x^2+9} + 2 \int \frac{1}{x^2+9} dx - 18 \int \frac{1}{(x^2+9)^2} dx = \int \frac{1}{x^2+9} dx$$

(do some algebraic re-arranging...)

$$\Rightarrow \int \frac{1}{(x^2+9)^2} dx = \frac{1}{18} \left[\frac{x}{x^2+9} + \int \frac{1}{x^2+9} dx \right]$$

$$= \frac{1}{18} \left[\frac{x}{x^2+9} + \frac{1}{3} \arctan\left(\frac{x}{3}\right) \right] + C$$

$$= \boxed{\frac{1}{18} \frac{x}{x^2+9} + \frac{1}{18 \cdot 3} \arctan\left(\frac{x}{3}\right) + C}$$

$\therefore$

10/6/09 (cont.)

NOTE: a table of integrals could also have been used, noting the following entries:

$$\bullet \int \frac{x}{(x^2+a^2)^2} dx = -\frac{1}{2(x^2+a^2)} + C$$

$$\bullet \int \frac{1}{(x^2+a^2)^2} dx = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\bullet \int \frac{1}{(x^2+a^2)^n} dx \quad (\text{where } n \text{ is a positive integer})$$

$$= \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{(2n-3)a^2} \int \frac{dx}{(x^2+a^2)^{n-1}}$$

→ this last one is an iterative formula (i.e. you keep applying the rule until things have been sufficiently simplified), but we could have derived it using integration by parts (as we did for P3); for example, consider

to find $\int \frac{1}{(x^2+1)^3} dx$, try using IBPs of $\int \frac{1}{(x^2+1)^2} dx$

$$\int \frac{1}{(x^2+1)^2} dx \quad \begin{array}{l} u = (x^2+1)^{-2} \rightarrow u' = -4x(x^2+1)^{-3} \\ v' = 1 \quad \quad \quad v = x \end{array}$$

$$= \frac{x}{(x^2+1)^2} + \int \frac{4x^2}{(x^2+1)^3} dx$$

$$= \frac{x}{(x^2+1)^2} + 4 \int \frac{(x^2+1)-1}{(x^2+1)^3} dx$$

$$= \frac{x}{(x^2+1)^2} + 4 \int \frac{1}{(x^2+1)^2} dx - 4 \int \frac{1}{(x^2+1)^3} dx$$

now rearrange

$$\rightarrow \int \frac{1}{(x^2+1)^3} dx = \frac{1}{4} \left[\frac{x}{(x^2+1)^2} + 3 \int \frac{1}{(x^2+1)^2} dx \right] + C$$

↳ which we knew how to solve from before