

REMINDER: Exam II next Wednesday (10/14)
(overview/review outline to be posted on class website later this week)

Quick Review

- $\int \frac{dx}{x^2-1} \rightarrow$ partial fractions, substitution
- $\int \frac{dx}{x^2+1} \rightarrow$ trigonometric substit.
- $\int \frac{dx}{(x^2+1)^2} \rightarrow$??? (will cover in class on 10/6)

TRIG. SUBSTITUTIONS

$\rightarrow$ use interrelationships (e.g. $\sin^2 \theta + \cos^2 \theta = 1, \dots$)
so to algebraically simplify the integrand

let $x = \tan \theta \rightarrow \int \frac{1}{1+x^2} dx = \int d\theta = \theta + C$
($dx = \frac{1}{\cos^2 \theta} d\theta$)

$$\int \frac{1}{1+x^2} dx = \int \frac{1}{1+\tan^2 \theta} \cdot \frac{1}{\cos^2 \theta} d\theta$$

$$= \int \frac{1}{\frac{1}{\cos^2 \theta}} \cdot \frac{1}{\cos^2 \theta} d\theta = \int d\theta = \theta + C$$

$$= \arctan(x) + C$$

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \frac{1}{\cos^2 \theta} \end{aligned}$$

10/5/09
Pg. 2

ex solve $\frac{dy}{dx} = \frac{1}{x^2 - 6x + 13}$

□ separate variables + integrate: $y(x) = \int \frac{1}{x^2 - 6x + 13} dx$

□ complete the square: $x^2 - 6x + 13 = x^2 - 6x + 9 + 4$
 $= (x-3)^2 + 4$
 $= (x-3)^2 + 2^2$

□ make a substitution: $\int \frac{1}{(x-3)^2 + 4} dx$
 $= \int \frac{1}{u^2 + 4} du \quad (\text{where } u = x-3, du = dx)$

□ make a (trig.) substitution: $u = 2 \tan \theta$
 $du = \frac{2}{\cos^2 \theta} d\theta$

$$\int \frac{1}{u^2 + 4} du = \int \frac{1}{4 \tan^2 \theta + 4} \cdot \frac{2}{\cos^2 \theta} d\theta$$

$$= \frac{1}{2} \int d\theta = \cancel{\frac{1}{2} \arctan\left(\frac{u}{2}\right) + C}$$

$$= \cancel{\frac{1}{2} \arctan\left(\frac{u}{2}\right) + C} = \frac{1}{2} \theta + C = \frac{1}{2} \arctan\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2} \arctan\left(\frac{x-3}{2}\right) + C$$

∴

□ what is the right substitution to make? Pg. 3

□ $a^2 + x^2, \sqrt{a^2 + x^2} \rightarrow x = a \tan \theta$

□ $\sqrt{a^2 - x^2} \rightarrow x = a \sin \theta$

ex $\int \frac{1}{\sqrt{9-x^2}} dx$ $x = 3 \sin \theta \rightarrow$ could also choose $x = 3 \cos \theta$
 $dx = 3 \cos \theta d\theta$
 $= \int \frac{1}{\sqrt{9-9\sin^2 \theta}} \cdot 3 \cos \theta d\theta$

$$= \int d\theta = \theta + c = \arcsin\left(\frac{x}{3}\right) + c$$

$\therefore$

NOTE: $\sqrt{\cos^2 \theta} = \begin{cases} \cos \theta & \text{if } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ -\cos \theta & \text{if } \theta < -\frac{\pi}{2} \text{ or } \theta > \frac{\pi}{2} \end{cases}$
 $\rightarrow$ need to be careful where your substitution is valid

~~ex~~ $\rightarrow$ what if we chose cos instead? $x = 3 \cos \theta$
 $dx = -3 \sin \theta d\theta$

$$\int \frac{1}{\sqrt{9-x^2}} dx = - \int d\theta = -\arccos\left(\frac{x}{3}\right) + c$$

$$= \arcsin\left(\frac{x}{3}\right) + c_0$$

★ NOTE: the consts. will be diff. between these two cases (we'll discuss this on 10/6)

ex $\int \frac{x^2}{\sqrt{9-x^2}} dx$

try $x = 3 \sin \theta$
 $dx = 3 \cos \theta d\theta$

$= \int \frac{x^2}{\sqrt{9-9\sin^2\theta}} \cdot 3 \cos \theta d\theta$

$= \int 9 \sin^2 \theta d\theta$

[use integration by parts]

$= 9 \left[-\frac{1}{2} \sin \theta \cos \theta + \frac{\theta}{2} \right] + C$

$= -\frac{9}{2} \frac{x}{3} \overset{\text{see aside}}{(\quad)} + \frac{9}{2} \arcsin\left(\frac{x}{3}\right) + C$

$= -\frac{1}{2} x \sqrt{9-x^2} + \frac{9}{2} \arcsin\left(\frac{x}{3}\right) + C$

need to be clear when you resubstitute,
 i.e. $\cos(\arcsin(\frac{x}{3}))$ is not so good,
 since $\cos \theta$ can be expressed more
 succinctly as a funct of x (see aside)

ASIDE

$\int \frac{x}{\sqrt{9-x^2}} dx$

$u = 9-x^2, du = -2x dx$
 $\rightarrow dx = -\frac{1}{2x} du$

$= \int \frac{x}{u^{1/2}} \cdot -\frac{1}{2x} du$

$= -\frac{1}{2} \int u^{-1/2} du$

$= -u^{1/2} + C$

$= -(9-x^2)^{1/2} + C$

ASIDE

$\sin^2 \theta + \cos^2 \theta = 1$

$\cos^2 = 1 - \sin^2 \theta$

$\cos \theta = \pm \sqrt{1 - \sin^2 \theta}$

$= \pm \sqrt{1 - (\frac{x}{3})^2}$

$= \frac{1}{3} \sqrt{9-x^2}$