

□ 'long division' refresher:

ex

simplify $\frac{x^5+3}{x^3+2x^2+5x-2} = x^2 - 2x - 1 + \frac{14x^2+x+1}{x^3+2x^2+5x-2}$

$$\begin{array}{r}
 x^2 - 2x - 1 \\
 x^3 + 2x^2 + 5x - 2 \overline{) x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 3} \\
 \underline{x^5 + 2x^4 + 5x^3 - 2x^2} \\
 -2x^4 - 5x^3 + 2x^2 + 0x + 3 \\
 \underline{-2x^4 - 4x^3 - 10x^2 + 4x + 0} \\
 -x^3 + 12x^2 - 4x + 3 \\
 \underline{-x^3 - 2x^2 - 5x + 2} \\
 14x^2 + x + 1
 \end{array}$$

ex solve the ODE

$$\frac{dy}{dx} = \frac{x^5+3}{x^3+2x^2+5x-2}$$

$$y(x) = \int \left[x^2 - 2x - 1 + \frac{14x^2+x+1}{x^3+2x^2+5x-2} \right] dx$$

still a hard integral to deal with potentially at this point

ex solve $\int \frac{3x^3-7x^2+17x-3}{x^2-2x+5} dx$

$$= \int \left[3x - 1 + \frac{2}{x^2-2x+5} \right] dx$$

$$= \frac{3}{2}x^2 - x + 2 \int \frac{1}{x^2-2x+5} dx$$

$$= \frac{3}{2}x^2 - x + 2 \int \frac{1}{(x-1)^2+4} dx$$

$$= \frac{3}{2}x^2 - x + 2 \left[\frac{1}{2} \arctan\left(\frac{x-1}{2}\right) \right] + C$$

$$\begin{array}{r}
 3x - 1 \\
 x^2 - 2x + 5 \overline{) 3x^3 - 7x^2 + 17x - 3} \\
 \underline{3x^3 - 6x^2 + 15x + 0} \\
 -x^2 + 2x - 3 \\
 \underline{-x^2 + 2x - 5} \\
 2
 \end{array}$$

ex solve $\int \frac{t^2+1}{t^2-1} dt$

$$= \int \left[1 + \frac{2}{t^2-1} \right] dt = t + 2 \int \frac{1}{t^2-1} dt$$

$$= t + 2 \int \frac{1}{(t+1)(t-1)} dt \quad \text{via the table of integrals}$$

$$= t + 2 \left[\frac{1}{2} (\ln|t-1| - \ln|t+1|) \right] + C$$

$$= t + \ln \left(\frac{|t-1|}{|t+1|} \right) + C$$

⇒ another method? (i.e. suppose we don't have a table, how could we still have solved the above?)

$$\square \frac{1}{x(x-1)} \neq \frac{1}{x} + \frac{1}{x-1}$$

rephrase: $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \quad (A, B - \text{consts.})$

$$1 = A(x-1) + Bx = x(A+B) - (1)(A)$$

$$x: 0 = A+B \rightarrow A = -B$$

$$\text{const: } 1 = -A \rightarrow A = -1, B = 1$$

$$\rightarrow \frac{1}{x(x-1)} = -\frac{1}{x} + \frac{1}{x-1}$$

Partial Fraction
Decomposition

ex solve $\frac{dy}{dx} = \frac{1}{x(x-1)}$

$$y(x) = \int \frac{1}{x(x-1)} dx = \int \left[-\frac{1}{x} + \frac{1}{x-1} \right] dx$$

$$= -\ln|x| + \ln|x-1| + C$$

$$= \ln \left| \frac{x-1}{x} \right| + C \quad \therefore$$

★ $\rightarrow$ to derive V-26, solve $\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$
[find A and B]

= 'Rules' for PFD (see later pg. of scanned notes here as well as pg. 334 in the H.H. book)

simplify
ex $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$$1 = Ax(x+1) + B(x+1) + Cx^2$$

$$x^2(0) + x(0) + 1(1) = x^2(A+C) + x(A+B) + (1)(B)$$

$$x^2: 0 = A+C \rightarrow C=1$$

$$x: 0 = A+B \rightarrow A=-1$$

$$\text{const: } 1 = B$$

$$\rightarrow \frac{1}{x^2(x+1)} = -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

9/30/09 (cont.)

ex solve $\int \frac{1}{x(x-1)} dx$

from before, we have: $= \int \left[-\frac{1}{x} + \frac{1}{x-1} \right] dx$

$$= -\int \frac{1}{x} dx + \int \frac{1}{x-1} dx = -\ln|x| + \ln|x-1| + C$$

$$= \ln \left| \frac{x-1}{x} \right| + C$$

'RULES' for Partial Fraction Decomposition (PFD)

see H.H.
book
Pg. 334

For a rational function $\frac{P(x)}{Q(x)}$

1 if $\text{degree}(P(x)) \geq \text{degree}(Q(x))$, try long division first and then PFD on the remainder

2 if $Q(x)$ has distinct linear factors, use $\frac{A}{x-c}$

3 if $Q(x)$ has repeated linear factors $(x-c)^n$, use

$$\frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \frac{A_3}{(x-c)^3} + \dots + \frac{A_n}{(x-c)^n}$$

4 if $Q(x)$ has an unfactorable quadratic $q(x)$, try $\frac{Ax+B}{q(x)}$ (or complete the square)

ex $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ (not $\frac{1}{x^2(x+1)} = \frac{A}{x^2} + \frac{B}{x+1}$!)

[repeated linear factors]

$$\rightarrow 1 = Ax(x+1) + B(x+1) + Cx^2$$

$$= x^2(C+A) + x(A+B) + (1)(B)$$

$$\begin{cases} x^2: 0 = C+A \\ x: 0 = A+B \\ \text{const: } 1 = B \end{cases}$$

$$\rightarrow \text{since } B=1, A=-1, C=1$$

$$\rightarrow \frac{1}{x^2(x+1)} = -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

ended
here on 9/30

9/30/09

ex) solve the ODE: $\frac{dy}{dx} = \frac{1}{x^2(x+1)}$

$$y(x) = \int \frac{1}{x^2(x+1)} dx = \int \left[-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right] dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

$$= \ln\left|\frac{x+1}{x}\right| - \frac{1}{x} + C$$

ex) solve
[repeated unfactorable
quadratic]

solve $\int \frac{2x^3 - x^2 + 2x - 2}{(x^2+2)(x^2+1)} dx$ [use Rule #4]

$$\frac{2x^3 - x^2 + 2x - 2}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1}$$

$$\rightarrow 2x^3 - x^2 + 2x - 2 = (Ax+B)(x^2+1) + (Cx+D)(x^2+2)$$

$$= Ax^3 + Ax + Bx^2 + B + Cx^3 + 2Cx + Dx^2 + 2D$$

$$= x^3(A+C) + x^2(B+D) + x(A+2C) + (B+2D)$$

(algebraically solve)

$$\circ A = 2-C, 2 = A+2C = 2-C+2C = 2+C$$

$$\rightarrow C=0, A=2$$

$$\circ B = -1-D, -2 = B+2D = -1-D+2D = -1+D$$

$$\rightarrow D=-1, B=0$$

$$\begin{array}{l} x^3: 2 = A+C \\ x^2: -1 = B+D \\ x: 2 = A+2C \\ \text{const: } -2 = B+2D \end{array}$$

four eqns.
four unknowns

$$\rightarrow \frac{2x^3 - x^2 + 2x - 2}{(x^2+2)(x^2+1)} = \frac{2x}{(x^2+2)} - \frac{1}{(x^2+1)}$$

$$\rightarrow = \int \frac{2x}{(x^2+2)} dx - \int \frac{1}{x^2+1} dx$$

$$= \int \frac{2x}{u} \cdot \frac{1}{2x} du - \int \frac{1}{x^2+1} dx$$

$$= \ln|u| - \arctan(x) + C = \ln|x^2+2| - \tan^{-1}(x) + C$$

(V-24)
solve using substitution and
table of integrals (for now):
let $u = x^2+2, du = 2x dx$
 $\rightarrow dx = \frac{1}{2x} du$