

9/22/09 (TUESDAY AFTERNOON)

IDEA: Given recent Matlab lab numerically estimating integrals via Riemann sums as well as geometric interpretation of the definite integral as an area under a curve, now seems like a natural time to motivate using integrals to estimate areas + volumes (H.H. ch. 8.1)

(starting point)

1 Area underneath a curve

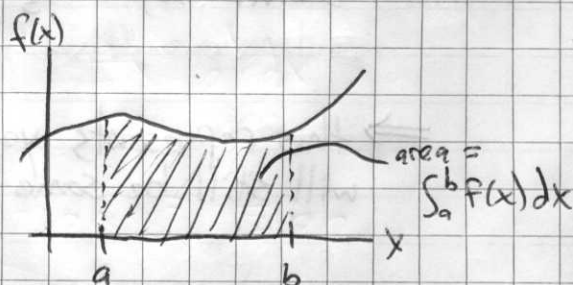
$$\text{area} = \int_a^b f(x) dx$$

$\underbrace{\hspace{1.5cm}}_{\text{sum of lots of little guys}} \underbrace{\hspace{1.5cm}}_{\text{height}} \underbrace{\hspace{1.5cm}}_{\text{width}}$

NOTE:

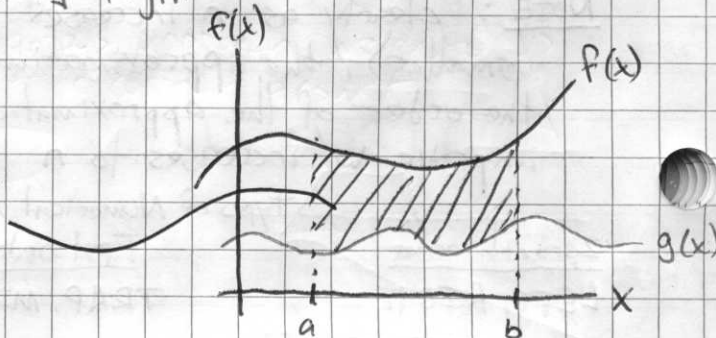
$$[f(x)] = \text{length}$$

$$[dx] = \text{length}$$

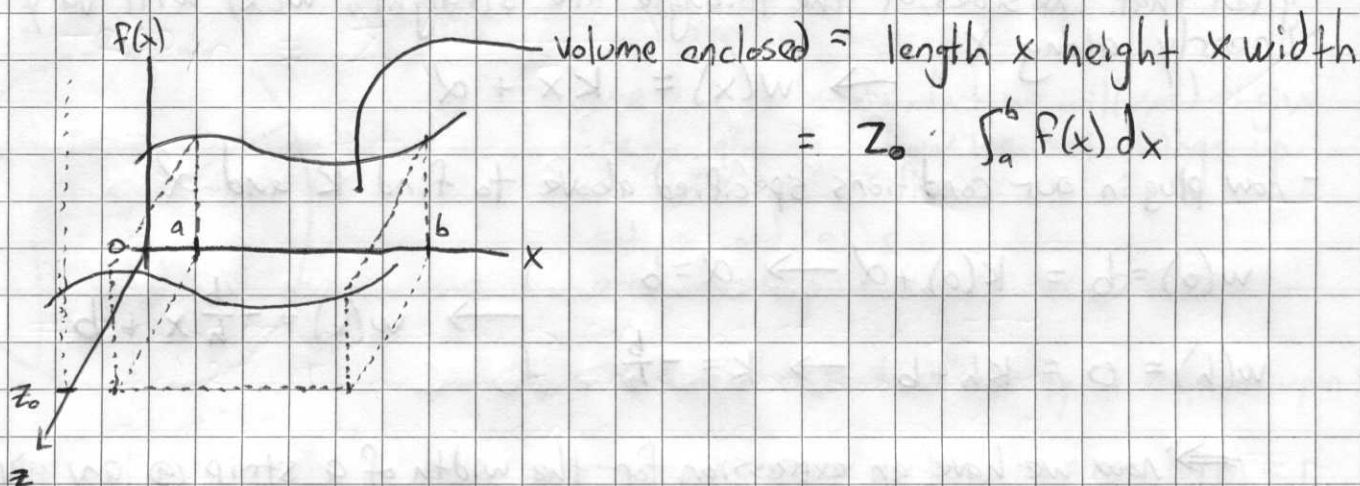


2 Area between two curves

$$\begin{aligned} \text{area} &= \int_a^b f(x) dx - \int_a^b g(x) dx \\ &= \int_a^b [f(x) - g(x)] dx \end{aligned}$$



3 Extending out into another dimension (i.e. area $\rightarrow$ volume)

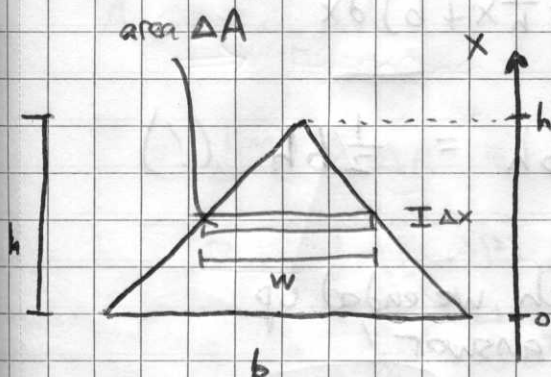


4 Arbitrary Geometric Shapes (2-D)

→ suppose you aren't given a nice curve $f(x)$ per se as in the previous example, but you need to set up the geometry yourself (e.g. determine a suitable coord. system)

ex find the area of a ~~right~~ isosceles triangle using Riemann sums
 [NOTE: we already know the answer! ; area = $\frac{1}{2}$ (base) $\times$ (height)]

(a) create a spatial axis (only one dimension needed here given the symmetry) and determine an orientation and scale)



(b) consider an infinitesimal strip of height Δx (where Δx is small such that the error due to using a rectangle across that particular slice is small)
 $\Delta A \equiv$ area of strip = width $\times \Delta x$ where width = $w(x)$

(c) consider the geometry such that you can specify $w(x)$ exactly

9/22/09

ex. (cont.)

- we know that $w(0) = b$ and $w(h) = 0$
- given that the sides of the triangle are straight, $w(x)$ will vary linearly along x
 $\rightarrow w(x) = kx + \alpha$

- now plug in our conditions specified above to find k and α

$$\left. \begin{array}{l} w(0) = b = k(0) + \alpha \rightarrow \alpha = b \\ w(h) = 0 = kh + b \rightarrow k = -\frac{b}{h} \end{array} \right\} \rightarrow w(x) = -\frac{b}{h}x + b$$

$\Rightarrow$ now we have an expression for the width of a strip @ any given height x and thus the area of the strip

$$\Delta A = w(x) \cdot \Delta x = \left(-\frac{b}{h}x + b\right) \Delta x$$

[d] now if we add up all the strips, we should have a good approximation of the area; in the limit where the height of the strip gets infinitesimally small (i.e. $\lim_{\Delta x \rightarrow 0}$) or put in other words, we have an infinite # of strips) this will become exact

$$A \simeq \sum_{x=0}^h \Delta A = \sum_{x=0}^h \left(-\frac{b}{h}x + b\right) \Delta x$$

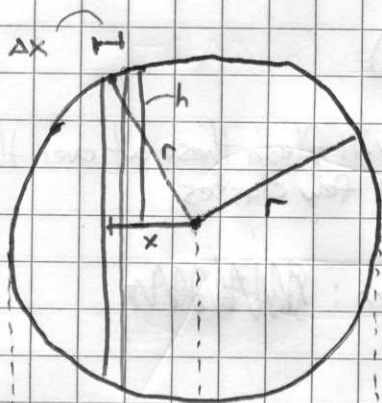
an integral we know how to solve!!

$$= \lim_{\Delta x \rightarrow 0} \sum_{x=0}^h \left(-\frac{b}{h}x + b\right) \Delta x = \int_0^h \left(-\frac{b}{h}x + b\right) dx$$

$$= -\frac{b}{h} \left(\frac{1}{2}x^2\right) \Big|_0^h + b(x) \Big|_0^h = -\frac{1}{2}bh + bh = \frac{1}{2}bh \quad (!)$$

$\Rightarrow$ though we took a more rigorous approach, we ended up w/ what we know to be the correct answer!

14 (cont.)

ex find the volume of a circle w/ radius r 

→ same plan as before, but we will need to give more thought as to how we set things up

- set up coord. axis
- area of strip: $\Delta A = \Delta x \cdot 2h$

→ but what is h ? clearly it depends upon x

→ we do know $h(-r) = h(r) = 0$, $h(0) = r$
[based upon our geometry set up]

⇒ consider the geometry (i.e. a right triangle), such that we must always have

$$x^2 + h^2 = r^2$$

thus $h = h(x) = \pm \sqrt{r^2 - x^2}$

→ thus the area of our strip becomes: $\Delta A = 2(r^2 - x^2)^{1/2} \Delta x$

- similar to before, we just add up all possible strips (by taking a limit)

$$A \approx \sum_{x=-r}^r \Delta A = \sum_{x=-r}^r 2(r^2 - x^2)^{1/2} \Delta x$$

$$= \lim_{\Delta x \rightarrow 0} \sum \Delta A = \int_{-r}^r 2(r^2 - x^2)^{1/2} dx$$

$$= 2 \int_{-r}^r \sqrt{r^2 [1 - (\frac{x}{r})^2]} dx = 2r \int_{-r}^r \sqrt{1 - (\frac{x}{r})^2} dx$$

$$= 2r \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 \theta} \cdot r \cos \theta d\theta$$

$$= 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta$$

(since $\sin^2 \theta + \cos^2 \theta = 1$, $1 - \sin^2 \theta = \cos^2 \theta$)

ok, so now how do we solve this integral?

→ we can rearrange our algebra a bit and make a trigonometric substitution

$$\text{let } \frac{x}{r} = \sin \theta$$

then $dx = r \cos \theta d\theta$
(limits change too!!)

$$\frac{x}{r} \Big|_{-r}^r = 1 - \sin \theta \rightarrow \theta = \pi/2$$

9/22/09

ex (cont.)

so we need to solve an integral of the form:

$$\int \cos^2 \theta d\theta$$

we could do this in a # of ways:

- 1) numerically (via your Matlab code!)
- 2) table of integrals
- 3) integration by parts

we will flesh these out over the next few classes

→ for the sake of time, we will simply take as given: ~~MAZDA~~

$$\int \cos^2 \theta d\theta = \frac{1}{2} \cos \theta \sin \theta + \frac{1}{2} \theta + C$$

→ back to our problem:

$$A = 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta = 2r^2 \left[\frac{1}{2} (\cos \theta \sin \theta + \theta) \right] \Big|_{-\pi/2}^{\pi/2}$$

$$= r^2 \left[\cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2} - \left(\cos\left(-\frac{\pi}{2}\right) \sin\left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right) \right) \right]$$

$$= r^2 \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right] = \pi r^2 (!)$$

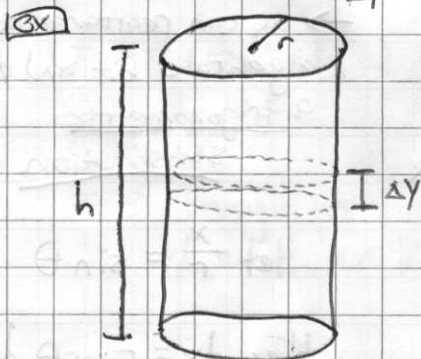
→ again, once the smoke cleared, we ended up w/ an answer we know we should end up with!

∴

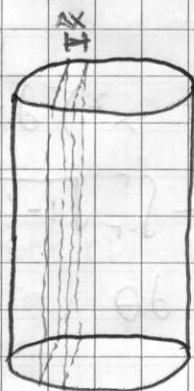
5 Volumes

→ now apply these same ideas to volumes, taking 'slices' rather than 'strips'

cylinder

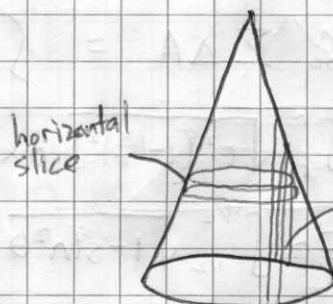


consider horizontal slices... (easy)



...or vertical slices (harder, but similar to approach we did above)

cone



or a vertical slice (both have to lead to the same answer!)

→ strategize to choose the easiest 'slicing paradigm'