

Math 256A

9/21/09

P3.1

- Project I due today
- Methods of Integration

Motivation: ODEs $\frac{dy}{dx} = f(x, y)$

$$\frac{dy}{dx} = f(x), g(x), h(x)j(y)$$

Solve via separation of variables, but still boils down to an integration problem

$$\frac{dy}{dx} = f(x) \rightarrow y(x) = \int f(x) dx$$

(thus solving an ODE can be thought of as an integration problem)

[Ch. 7.1] Chain Rule in Reverse $\Leftrightarrow$ 'Substitution'

$$\text{chain rule: } \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$\text{let } u = g(x), \quad du = g'(x) dx$$

$$\int f[g(x)] g'(x) dx = \int f(u) du$$

another point of view

$$\int dF[g(x)] = F[g(x)]$$

$$= \int F'[g(x)] g'(x) dx \Rightarrow \text{chain rule in reverse!}$$

linear ODE

linear ODE:

$$y' + f(x)y = g(x)$$

ex

$$\frac{dy}{dx} = (2x+1)e^{x^2+x}$$

$$y(x) = \int (2x+1)e^{x^2+x} dx$$

~~⊕~~ ~~⊖~~ ~~⊗~~ ~~⊙~~ $\Omega = x^2 + x$, $d\Omega = (2x+1) dx$

$$\rightarrow dx = \frac{d\Omega}{2x+1}$$

$$\begin{aligned} \int (2x+1)e^{x^2+x} dx &= \int (2x+1)e^{\Omega} \frac{d\Omega}{2x+1} \\ &= \int e^{\Omega} d\Omega = e^{\Omega} + C = e^{x^2+x} + C \therefore \end{aligned}$$

ex

$$\int \frac{1}{x \ln x} dx$$

$$\begin{aligned} w &= \ln x \\ dw &= \frac{1}{x} dx \\ \rightarrow dx &= x dw \end{aligned}$$

$$\begin{aligned} u &= x \\ du &= dx \\ \int \frac{1}{x \ln x} dx &= \int \frac{1}{u \ln u} du \\ &= \int \frac{1}{u \ln u} du \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x \ln x} dx &= \int \frac{1}{x w} x dw = \int \frac{1}{w} dw \\ &= \ln|w| + C = \ln|\ln x| + C \therefore \end{aligned}$$

Generic Examples

$$\boxed{A} \int g'(x) e^{g(x)} dx = e^{g(x)} + C$$

$$\boxed{B} \int \frac{g'(x)}{g(x)} dx = \ln |g(x)| + C$$

$$\boxed{\text{ex}} \int 4x \sqrt{x^2+1} dx$$

$$\begin{aligned} w &= x^2 + 1 \\ dw &= 2x dx \\ \rightarrow dx &= \frac{1}{2x} dw \end{aligned}$$

$$= \int 4x \sqrt{w} \frac{1}{2x} dw$$

$$= \int 2w^{1/2} dw = 2 \left(\frac{2}{3} w^{3/2} \right) + C$$

$$= \frac{4}{3} w^{3/2} + C = \frac{4}{3} (x^2+1)^{3/2} + C \quad \therefore$$

$$\boxed{\text{ex}} \int \tan(x) dx \quad (\text{over intervals where } -\frac{\pi}{2} < x < \frac{\pi}{2})$$

$$= \int \frac{\sin(x)}{\cos(x)} dx$$

$$\begin{aligned} u &= \cos(x), \quad du = -\sin(x) dx \\ \rightarrow dx &= -\frac{1}{\sin(x)} du \end{aligned}$$

$$= \int \frac{\sin(x)}{u} \cdot -\frac{1}{\sin(x)} du$$

$$= -\int \frac{1}{u} du = -\ln |u| + C = -\ln |\cos(x)| + C \quad \therefore$$

$$\boxed{\text{ex}} \int x \sqrt{2x-1} dx$$

$$\begin{aligned} \phi &= 2x-1, \quad d\phi = 2 dx \\ \rightarrow dx &= \frac{1}{2} d\phi \end{aligned}$$

$$= \int x \phi^{1/2} \frac{1}{2} d\phi$$

$$\rightarrow x = \frac{1}{2} (\phi+1)$$

$$= \frac{1}{2} \int \frac{1}{2} (\phi+1) \phi^{1/2} d\phi = \frac{1}{4} \int [\phi^{3/2} + \phi^{1/2}] d\phi$$

$$= \frac{1}{4} \left[\frac{2}{5} \phi^{5/2} + \frac{2}{3} \phi^{3/2} \right] + C$$

$$= \frac{1}{10} (2x-1)^{5/2} + \frac{1}{6} (2x-1)^{3/2} + C$$

Definite Integrals

→ don't forget to change the limits!

ex $\int_0^4 2x \sqrt{x^2+1} dx$

let $u = x^2+1 \rightarrow dx = \frac{1}{2x} du$

$$= \int_1^{17} u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_1^{17} = \frac{2}{3} [17^{3/2} - 1]$$

$$\int_0^4 2x \sqrt{x^2+1} dx = \int_0^4 u^{1/2} du \quad \times$$

$$= \int u^{1/2} du \quad \times$$

$$\rightarrow \int_1^{17} u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_1^{17} = \frac{2}{3} (x^2+1)^{3/2} \Big|_0^4$$

ex $\int_{1/2}^1 \frac{1}{x^2} e^{1/x} dx$

$u = \frac{1}{x}$

→ $e^2 - e$