

□ Tuesday afternoon will discuss material covered up to this point to help give big picture for 9/16 Exam I

Topics that ~~might~~ have been discussed:

- basic nature of ODEs and contexts they might be used in
- identifying types of ODEs
- basic review of calculus (e.g. differentiating functions)
- translating 'word problems'
- how to solve ODEs of the form: $\frac{dy}{dx} = f(x)$
- " autonomous " $\frac{dy}{dx} = g(y)$ } → integrate!!

→ should know how to solve autonomous 1st order ODE: $\frac{dy}{dt} = ay + b$
($a, b = \text{const.}$)

- determining whether a given solution satisfies a given ODE
- translating an ODE into a qualitative picture
- Riemann sums
(LEFT, RIGHT, MID, TRAP)

Archimedes of Syracuse (~200 BC)

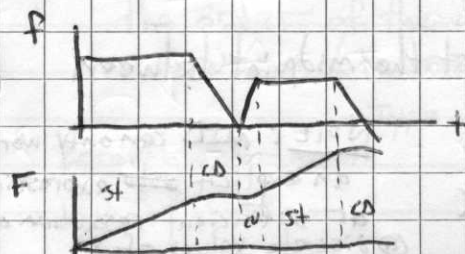
student of Euclid; method of quadratures to find areas (but could only be used in cases w/ high degree of symmetry)

Riemann Sums

□ allows one to numerically approximate the value of an integral

→ though note that in the limit your 'rectangles' get infinitesimally small, it approaches the exact value
(this was the context upon which Riemann developed the notion around ~mid-1800's)

□ consider an 'area underneath a curve'



st - straight
(const. slope)

cu - concave up

cd - concave down

→ 'second Fundamental theorem of calc.'

$$\int_a^x F(t) dt = F(x)$$

where F is cont. on $t \in [a, x]$ and $\frac{d}{dt} F(t) = f(t)$

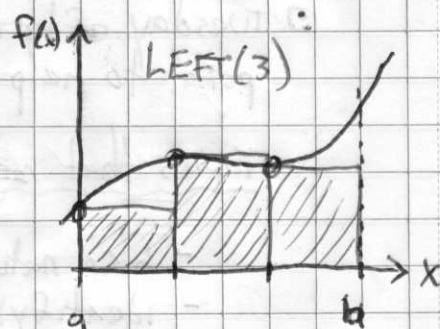
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□ Review: LEFT and RIGHT Riemann Sums

$$I \equiv \int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f(x_i) \Delta x$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

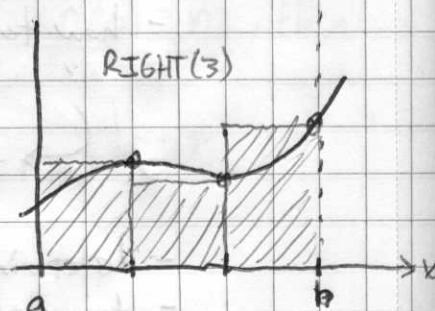
where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$



→ can't practically take the limit $\lim_{n \rightarrow \infty}$, so in the finite case, we have:

$$\text{LEFT}(n) = \sum_{i=0}^{n-1} f(x_i) \Delta x$$

$$\text{RIGHT}(n) = \sum_{i=1}^n f(x_i) \Delta x$$



□ depending upon the nature of $f(x)$ over the interval of integration, our Riemann sums can either under- or over-estimate (or both!) since the rectangles can ~~miss~~ exclude/include relevant areas

→ thus important to know whether our estimate is too high or too low

ex. if $f(x)$ is strictly increasing on $[a, b]$, LEFT will under-estimate while RIGHT will over-estimate

⇒ so, can we do better?

□ what if the heights of our rectangles either take both ends into account (TRAP) or some intermediate value on the interval (MID)?

Midpoint Rule: consider evaluating $f(x)$ at the midpoint between x_i and x_{i+1}

$$\text{MID}(n) \equiv \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x$$

NOTE: MID can only work if you have an explicit ~~set~~ expression for $f(x)$ or sufficient resolution of $f(x)$'s sampled discrete values of x

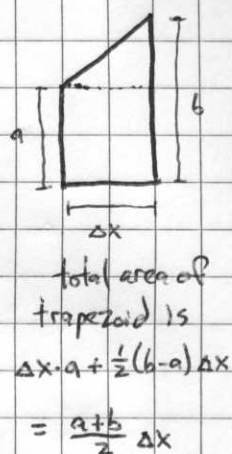
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Trapezoid Rule: Approximate the area under $f(x)$ between x_i and x_{i+1} using a corresponding trapezoid

$$\text{TRAP}(n) \equiv \sum_{i=0}^{n-1} \left(\frac{f(x_i) + f(x_{i+1})}{2} \right) \Delta x$$

→ can readily show that $\text{TRAP}(n) = \frac{1}{2} [\text{LEFT}(n) + \text{RIGHT}(n)]$

⇒ these refined estimates can do better than LEFT and RIGHT, but there can still be some degree of error and thus we might under/over-estimate



[see H.H. 7.5 ^{7.6 too} for a nice (visual) discussion of how to think about this]

Simpson's Rule

(we will not derive here, but the basic idea is to use a quadratic function that matches $f(x_i)$ and $f(x_{i+1})$ so to capture a degree of the curvature in $f(x)$; we call this a higher order approximation)

$$\text{SIMP}(n) = \frac{1}{3} [2 \text{MID}(n) + \text{TRAP}(n)]$$

⇒ this represents yet an even better approx. (but note that there will still be some degree of error)

NOTE: clearly as n increases (i.e. the width of our rectangles get smaller), the approximations will improve; however, the higher the order of the approximation, the more quickly things will improve w/ increases to n

Types of Numerical Approxs. to Integrals

Zeroth order
LEFT, RIGHT

First order
TRAP, MID

Second order
SIMP

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