

1. Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called continuous compounding.) Suppose an initial deposit of d dollars is made.

(a) Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .

(b) Suppose withdrawals are made at a constant rate w . Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .

'grows'

'rate proportional to the balance at that time'

$$\boxed{a} \quad \frac{dx}{dt} = kx$$

↑
const. of
proportionality

$$x(0) = d \quad [\text{dollars}]$$

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initial condition

$$\boxed{b} \quad \frac{dx}{dt} = kx - w$$

↑
withdrawal
rate

$$x(0) = d$$

20. In biology, allometry is the study of the relative size and growth of different parts of an organism. Suppose $x = x(t)$ and $y = y(t)$ are measures of the sizes of two different parts of a particular organism as functions of time. A simple model of allometry is based on the assumption that the per unit growth rates of the different parts are proportional to each other. Treating y as a function of x , derive a differential equation for y . Is this equation linear or non-linear?
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'per unit growth rate'

↪ looking for $\frac{dy}{dx}$

Assumption (given): $k \frac{x'}{x} = \frac{y'}{y}$ (per unit growth rates are proportional)

$$\frac{dy(x)}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} \cdot x' = y'$$

$$\text{but } \frac{y'}{x'} = k \frac{y}{x} = \frac{dy}{dx}$$

$$\rightarrow \boxed{\frac{dy}{dx} = k \frac{y}{x}}$$

7. A basic model of the growth of a tumor is based upon the assumption that the per unit volume growth rate of the tumor decreases exponentially with time (i.e., proportionally to an exponential of the form e^{-bt} for a constant $b > 0$). Assume the tumor initially has volume $v_0 > 0$. Write a differential equation and initial condition for the tumor volume $v = v(t)$ as a function of time.
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EXERCISE I.2.7. By assumption the per unit growth rate v'/v is proportional to e^{-bt} so we write $v'/v = ae^{-bt}$ for a constant of proportionality a . We assume $a > 0$ to model tumor growth. The initial size of the tumor is v_0 so we have the equations

$$\begin{aligned}v' &= ae^{-bt}v \\v(0) &= v_0.\end{aligned}$$

9. Newton's law of cooling states that the rate at which a body cools is proportional to the difference in temperature between the body and its surrounding environment. Write a differential equation for the temperature of the body.
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EXERCISE I.2.9. Let $x = x(t)$ be the temperature of the body. Then $x' = c(T_e - x)$ where c is the constant of proportionality and T_e is the temperature of the surrounding environment. The constant c must be positive so that the temperature of the body increases when it is less than the external temperature and decreases when it is greater.

11. Suppose two distinct cultural groups live in the same city. Each grows at a rate proportional to its numbers. However, each group loses population numbers at a rate proportional to the size of the other (since the two groups do not get along). Write a compartmental model (two differential equations) for the rate of change of each group's size. Is the system of differential equations linear or nonlinear?
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EXERCISE 1.2.11. Let $x = x(t)$ and $y = y(t)$ be the size of the two population groups as functions of time. Then

$$\begin{aligned}x' &= \text{inflow rate} - \text{outflow rate} \\ &= \text{growth rate} - \text{loss rate}.\end{aligned}$$

By assumption the growth rate is proportional to x and the loss rate is proportional to y . Thus, $x' = r_1x - ay$ for positive constants r_1 and a . Similar reasoning leads to an equation for y' :

$$\begin{aligned}x' &= r_1x - ay \\ y' &= r_2y - bx\end{aligned}$$

where r_1 , r_2 , a and b are positive constants. The system is linear.

12. Consider two populations that grow exponentially in the absence of the other (i.e., their per unit birth and death rates are constants). When placed together in a common habitat, the two populations compete for a vital resource. Because of this competition each population's per unit death rate is increased, but the per unit birth rates remain constant. Specifically, assume each population's per capita death rate is proportional to the other population's density. Write down a system of differential equations for the population densities x and y of each population. Is the system linear or nonlinear? (This model is a famous system in theoretical ecology called the Lotka-Volterra competition model.)
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EXERCISE 1.2.12. From the balance law

$$\begin{aligned}x' &= \text{inflow rate} - \text{outflow rate} \\ &= \text{birth rate} - \text{death rate}\end{aligned}$$

and the given assumptions that the per unit birth rate is constant and per unit the death rate is proportional to y we have $x'/x = b_1 - cy$ or $x' = b_1x - cxy$. Similar reasoning leads to an equation for y' :

$$\begin{aligned}x' &= (b_1 - c_1y)x \\ y' &= (b_2 - c_2x)y\end{aligned}$$

where all (coefficients) parameters are positive. The system is nonlinear.

18. Consider two armies engaged in a battle. Let x and y represent the strengths of the two armies (measured, for example, as the number of troops and/or armaments). Assume the strength of each army decreases at a rate proportional to the strength of the other army. In addition, each army receives reinforcements at a constant rate. Write differential equations for x and y . Are these equations linear or nonlinear?
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EXERCISE I.2.18. The balance law

$$\begin{aligned}x' &= \text{inflow rate} - \text{outflow rate} \\ &= \text{reinforcement rate} - \text{decrease rate}\end{aligned}$$

and the assumptions

$$\text{reinforcement rate} = \text{constant} = b$$

$$\text{decrease rate} = ay$$

for positive constants a, b yields the equation $x' = b - ay + b$. Similar reasoning leads to an equation for y :

$$x' = -ay + b$$

$$y' = -cx + d$$

where a, b, c and d are positive constants. The system is linear.
