

SOLUTIONS

Group 1

$$\frac{dx}{dt} = t(1+x)$$

Solution:

$$x(t) = Ce^{t^2/2} - 1$$

Group 2

$$\frac{dx}{d\phi} = ax + \cos(b\phi)$$

Solution:

$$x(\phi) = \begin{cases} Ce^{at} + \frac{1}{a^2+b^2} [b \sin(b\phi) - a \cos(b\phi)] & \text{if } a^2 + b^2 \neq 0 \\ \phi + C & \text{if } a = b = 0 \end{cases}$$

Group 3

Suppose a population $x = x(t)$ naturally grows at an exponential rate r , i.e., $x' = rx$. However, suppose the population is also subject to *harvesting* (removal of individuals) at a constant rate $h > 0$. Then $x' = rx - h$. Find a solution to the initial value problem given $x(0) = x_o$.

Solution:

$$x(t) = \left(x_o - \frac{h}{r}\right) e^{rt} + \frac{h}{r}$$

Group 4

$$\xi \frac{d\theta}{d\xi} = -\theta + \xi^{1/3}$$

Solution:

$$\theta(\xi) = \frac{c}{\xi} + \frac{3}{4}\xi^{1/3}$$

Group 5

$$\frac{dp}{dx} = p \cos(x) + \sin(x)$$

Solution:

$$p(x) = Ce^{\sin x} + e^{\sin x} \int e^{-\sin x} \sin x \, dx$$

Group 6

$$\frac{dy}{dx} = -\frac{y}{x} + \frac{2}{1+x^2} \quad y(1) = \ln 8$$

Solution:

$$y(x) = \frac{\ln 4}{|x|} + \frac{1}{x} \ln(1+x^2)$$

Bonus

Consider the following ODE:

$$\frac{dx}{dt} = \frac{x}{t} + te^{-t}$$

with the initial condition $x(-1) = 0$.

- Can you foresee anything potentially problematic?
- Find an explicit solution to the ODE.
- Compute a numerical solution (e.g., dfield).
- What do you notice about the *problem*?
- How does the Existence and Uniqueness Theorem come into play here?

Solution: The right-hand side of the ODE is not continuous for $t = 0$, so we might anticipate a problem there. Numerically, solutions seem to approach the origin from the initial condition as $t \rightarrow 0^-$. Solving the ODE analytically, we have

$$x(t) = -e|t| - te^{-t}$$

and $\lim_{t \rightarrow 0^-} x(t) = 0$, consistent with the numerical solution. Given the discontinuity at $t = 0$, we are unsure if there will be a unique solution when $t = 0$. Likely not, given that different initial conditions (provided $t < 0$) will also apparently lead to $\lim_{t \rightarrow 0^-} x(t) = 0$.
