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Introduction

I have been a mathematics teacher for nearly 11 years. I started as a graduate student instructor for a pre-algebra course back in 1997, and I have just finished teaching my twentieth college-level mathematics course. When I focus on the years in between, what comes to mind is not the spectrum of topics and levels covered, but rather the profound, significant evolution of my personal conception of what it means to be at the head of a mathematics class.

When I first began teaching, I believed the burden of ensuring a successful learning experience lied exclusively upon the instructor. I equated teaching to delivering thorough, comprehensive lectures that conceptually covered all relevant topics, cases, and exceptions. My explanations contained very well elaborated mathematical insights; but they were the product of my own careful thinking, not the product of my students' work. But when my very own calculus students, responding to my very own mid-semester feedback request, suggested my increasingly perfect lectures where not quite so ideal, I was called to reconsider. Since then, numerous individuals and experiences have inexorably contributed to shaping the view of mathematics teaching I hold today.

This view rests squarely upon the fundamental belief that the ideal teacher is a leader who expertly facilitates rather than delivers. This means the ideal teacher applies her own expert knowledge of mathematics to facilitating the production of student-generated mathematics. The ideal teacher understands that knowing mathematics means doing mathematics, and that this *doing* leads to a sense of *ownership* over the final product that cannot be acquired from somebody else but can only be self-generated.

In what follows I draw on specific examples from my teaching experience to illustrate how my quest for becoming the best possible mathematics teacher has been re-defined and re-directed by this new personal view.

My teaching philosophy in context

1. How I center the learning on the students: establishing my role as expert facilitator

   On one occasion, while grading homework, I noticed a large and varied collection of answers for a problem involving arithmetic sequences. Some solutions portrayed incoherent procedures that somehow lead to the correct answer; a few featured extremely creative and correct solutions; others were structurally correct, but lacked appropriate justification. What I saw here was an opportunity to use student-generated work as a springboard to a student-centered whole class discussion. I hand-wrote a representative sample of the answers and made copies. The next day, the lesson began with a mock grading exercise for the students. If these were their best friend's answers, could they grade them fairly? Were the arguments and answers correct or incorrect?

   This activity engaged the class in the non-trivial process of reflecting upon their own work and deciding upon the validity of peer-generated solutions. The class got a good sense of why writing what one means and using proper notation are key practices for communicating ideas, even among peers. The task also demonstrated to students that it is possible to give creative, one-of-a-kind arguments leading to a correct solutions, and that such non-routine reasoning must be carefully analyzed before flagged as correct or incorrect.

   I have recently started using another form of whole-class *collaborative inquiry* in my classes. The logistics are as follows. Before the lesson starts, I divide all the blackboards in the room (at least two, ideally) into sections. At the top of each section, I write a different problem, a total of six to nine. The students, organized in small groups, pick a problem for their group to work on, and begin working on it directly on the board. As a group finishes working on a problem they move onto the next one. If a solution for this second problem has already been written up, the task becomes not solving, but rather evaluating the solution that already appears on the blackboard. The group that evaluates may make corrections (without erasing), add work that clarifies or explains, and even write an alternate solution. At the end of 20 to 40 minutes, all the groups have covered all the boards.

   My task now becomes facilitating a whole-class discussion in which the students will eventually determine the correctness of their own boardwork and, in the process, notice relations among questions, ideas, specific solution strategies, and broader topics. As with the peer-grading activity, the students are themselves engaged in doing mathematics, not just by solving problems but also by reflecting upon and evaluating their own work.

2. How I manage classroom discourse and collaborative inquiry: leading through motivation and empowerment

   I have just described the structure of two particular forms of collaborative inquiry I use in my courses. But what are some practical ways of generating a whole-class discussion that is conceptually focused and keeps the process of doing mathematics centered on the students? An effective approach is to start by asking students to share their own opinions about the correctness of the work they have produced. If students have just finished working on any kind of non-routine, conceptual problem, such a request is not only natural but highly anticipated by the students: they want to share the arguments/partial arguments they have thought about and arrive at a definite validation of whatever the correct answer to a question might be.

   As the facilitator, I mainly ask and re-direct questions. Here is an example from my most recent course, *Mathematics from a historic perspective*, a course taken by a good number of secondary education and mathematics education majors. One of the questions groups had worked on on the board during the collaborative inquiry exercise read:

   > "Prove or disprove: $0.9999\cdots = 1$."

   This identity had actually appeared as a true fact in the reading for that day's lesson, which centered about Cantor's proof of the non-denumerability of the real numbers. The identity was built upon, but not explicitly justified in the reading. The proof of the identity I had in mind was actually up on the board, written by the team of a the student who had posed the question to me during office hours. After that team explained their board work, two questions were directed at me from another table:

   > "Is that right?...I mean, is that how you want us to answer that question?"

   I responded that, as they knew, this was not about what I wanted, but rather about what was true or not true mathematically speaking. So I re-directed the question to the whole class. Were the written solution and the argument that was given to back it up correct or not? A new student now mentions that he is also unsure about the approach, and proposes re-casting the problem in terms of sequences, a topic discussed in class about two weeks before. Surprisingly, I had not thought about this sequence approach myself, but I realize his suggestion holds a great opportunity for bridging two main course topics using student-generated mathematics. So, the student who had come forth with the idea, leads the class into fleshing out the details of the argument for a proof. In the midst of this process both the student and the rest of the class realize that the key for completing the argument is seeing the geometric sequence built within it. This "Aha!" moment allows them to finish the proof. The board ends up bearing two alternative, student-validated solutions to an insightful non-routine problem that I had *never* formally discussed in class.

3. How I teach mathematics as though students have read the book: defining self-regulating practices

   According to Bob Megginson, Professor of Mathematics and LSA[1] Associate Dean at the University of Michigan, the first three rules for highly increasing the likelihood that students will read their books before coming to class are:

   (a) Teach as though students have read the book.
   (b) Teach as though students have read the book.
   (c) Teach as though students have read the book.

   Here are three practical ways in which I try to do just that, one for each of these rules.

   (a) *Give an in-class quiz on the reading before discussing the reading's material.*
   This is not so much about accountability but mostly about purpose. I usually give these quizzes at the beginning of class, and I use them as an intellectual warm-up for the day's lesson, which I typically start off by building conceptually on questions on the quiz. I have found this "reading-quiz" strategy works very well for helping student establish important self-regulating practices, particularly because my quizzes assume you had read the chapter with paper and pencil. In fact, often my "reading-quizzes" are open book and open notes, or can be solved in pairs. The students welcome this practice, but have learned that this means they should prepare for a challenging quiz. The focus, again, is on the quiz as an opportunity to learn from the reading, not to recall systematically-read facts.

[1]LSA stands for College of Literature, Science and the Arts.

(b) *Organize a whole-class board discussion on the reading's material.*

I have used this approach with a lot of success in mathematics and mathematics education classes alike. I think of this activity as a variant of the whole-class collaborative inquiry board-work discussed in Items (1) and (2) above. The idea is simply to select problems that can only be solved if one has done the reading. Invariably, the likelihood that at least one student in each team has read some of the chapter is relatively high. This is enough to get the process going. I then facilitate the discussion as I described in Item (2).

(c) *Design lessons that build upon material from the reading, but are not just the material in the reading.*

I design all of my lessons around questions and examples that build upon the concepts discussed in the day's reading, but that are non-trivially different from the questions and examples *in* the reading. What I consider "non-trivially different" depends on the context. In a mathematics class, this minimally means doing more than changing the numbers on a problem, for example. In a mathematics education class for elementary teachers, this might mean discussing what constitutes 1/3 of a whole consisting of three different-size pizzas, when the reading discussed a pictorial representation of 1/4 of a whole consisting of three (equal) cups of flour.

4. Embracing the "tool-box metaphor": how knowing mathematics means being able to do mathematics

My teaching philosophy and methods place emphasis on learning mathematics by doing mathematics and not on being able to systematically master some well-defined set of procedures. This means that my students often develop a need to re-define how they measure what they know. This becomes particularly relevant to them as they set out to prepare for an exam. Will I ask a question like "Is $0.9999\cdots = 1$?" in the exam or not? How are they supposed to study? How can they be sure they are well prepared for a test?

I answer these kinds of questions by explaining what I call "the tool-box metaphor." Suppose we measure a craftsman's ability by his success in using the tools in his tool-box so as to tackle a problem that may not be exactly identical to any of the problems he has solved before. Then how can this craftsman prepare for the unknown problem? The focus must not be on *anticipating* the problem to come, but on introspecting as much as possible on how one used the tools to solve past problems. In short, the best way to prepare for one of my exams, is to keep solving a variety of problems, and thinking about what each of these problem teaches about the capabilities of the tools we actually have. I stress that the focus here is on *variety* of problems not on quantity.

The ultimate measure of "preparedness" lies in feeling a sense of ownership over the mathematics, a readiness to tackle the unknown problem to come. The same problem we saw in class may not be on the exam. But the kind of reasoning a student may have used for thinking about it will be represented on the exam in one form or another. The bottom line remains: knowing mathematics means being able to do mathematics.

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Just as with the expert craftsman, what defines a good facilitator is not what he has accomplished but how he approaches the ever changing contexts and challenges that may lie ahead. To grow and develop my facilitator role, I continue to introspect upon what past experiences have taught me. I deeply enjoy the learning process I have engaged in. It has served me well and, my students' feedback suggests many are experiencing the empowerment that comes from feeling a sense of ownership over the math they produce. This is my best reward; I am thankful for the opportunity to help others experience mathematics for what it truly is: a creative, deeply beautiful, forever in-the-making discipline.