

Related Rates

1. INTRODUCTION

One of the reasons why differentiation is important in, for example, physics and engineering, is that velocity is the first derivative of position with respect to time. In general, the derivatives represent rates of change. In this lecture, we look at problems involving at least two different rates of change, where at least one rate of change is known, and the other one we wish to determine. We can calculate the rate of change of one quantity by relating it to the other quantities whose rates of change are known.

2. METHOD

1. Draw a picture to help understand the problem.
2. Identify the known rates of change and the rate of change to be determined.
3. Find an equation (usually found through geometry) that relates all of the relevant quantities.
4. Differentiate both sides of this equation, usually with respect to time. This is called implicit differentiation. This will give a relationship between the known rates of change and the unknown rate of change.
5. Solve for the unknown rate of change.
6. Plug in the known rates of change and the known quantities.

3. PROBLEMS

Solve the following problems. Each problem is followed by a series of steps which will guide you through the process.

Problem 1A.

A 4-meter ladder stands against a wall. The foot of the ladder moves outward at a speed of 0.1 m/sec when the foot is 2 m away from the wall. At that moment, how fast is the top of the ladder falling?

Problem 1B.

If the foot of the ladder moves out at a constant speed, does the speed at which the top falls increase or decrease?

Problem 2.

A water tank is in the shape of an inverted cone with height 20 m and top radius 4 m. Water is flowing into the tank at a constant rate of $0.1 \text{ m}^3/\text{min}$.

A. How fast is the water level in the tank rising at the moment that the depth of the water is 5 m?

B. Now, alter the problem such that the water is also leaking out of the tank at a rate of $2.5 \times 10^{-4}h^2 \text{ m}^3/\text{min}$, where h is the depth of the water in the tank in meters. Can the tank ever overflow?

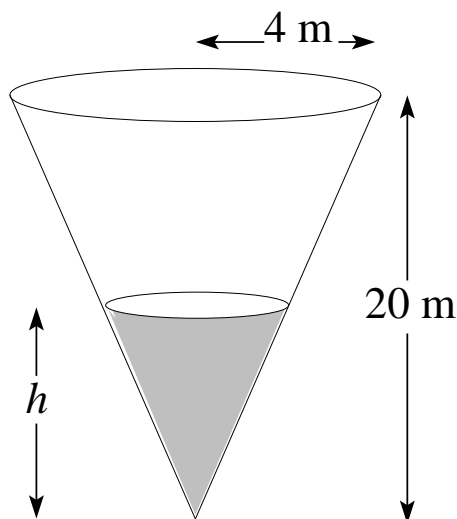


FIGURE 1. The set-up for Problem 2

Hints:

Problem 1A.

Step 1.

Define the distance of the foot of the ladder from the wall to be x meters and the distance of the top of the ladder from the ground to be y meters. Since these distances are changing with time, we think of x and y as functions of time, t . In this problem, we are asked to find the rate at which the top of the ladder is falling in the moment at which $x = 2$ and the ladder moves outward at a speed of 0.1 meter/sec. This is $\frac{dy}{dt}$. The rate at which the foot of the ladder moves outward is given by $\frac{dx}{dt}$. What is the value of $\frac{dx}{dt}$?

Step 2.

Use the Pythagorean Theorem to find an equation relating x and y .

Step 3.

Differentiate this implicit relation with respect to t , remembering that both x and y are functions of t . We obtain an equation showing the relationship between the derivatives.

Step 4.

Solve for $\frac{dy}{dt}$.

Step 5.

Use the equation in step 2 to find the value for y when $x = 2$. Now substitute the known values into the expression for $\frac{dy}{dt}$. Give units.

Problem 1B.

Consider the equation we found in step 4. Recall that speed is the magnitude of the velocity.

Problem 2A. **Step 1.**

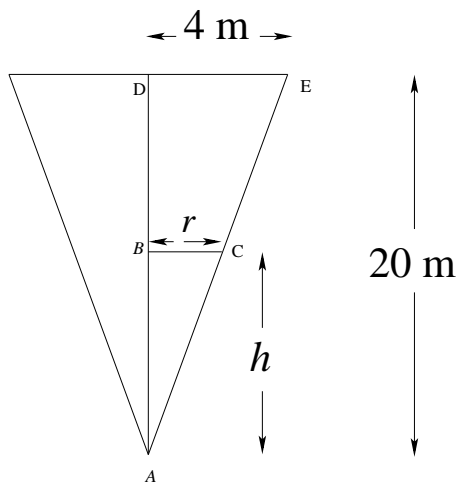


FIGURE 2. The set-up for Problem 2

The water in the tank forms a smaller cone with the same angles as the tank. Denote the depth of the water in the tank by h and the radius of the smaller cone with that height by r . Denote the volume of water by V . Now h , r , and V are all functions of time, t . The rate of change of the volume of water is given by $\frac{dV}{dt}$, and the rate of change of the depth of the water in the tank is given by $\frac{dh}{dt}$. The object of this problem is to find $\frac{dh}{dt}$ in the moment at which $h = 5$ m and water is flowing into the tank at a constant rate of $0.1 \text{ m}^3/\text{min}$. What is the value of $\frac{dV}{dt}$?

Step 2.

The two triangles, $\triangle ABC$ and $\triangle ADE$ are similar. Use this fact to find an expression for the radius r in terms of the height h .

Step 3.

The volume of a cone with radius r and height h is given by,

$$V = \frac{\pi}{3}r^2h.$$

Now, use your answer in step 2 to find an expression for the volume of water as a function of h .

Step 4.

Differentiate this expression with respect to t , remembering that h is a function of t . We obtain an equation that shows the relationship between $\frac{dV}{dt}$ and $\frac{dh}{dt}$.

Step 5.

Solve for $\frac{dh}{dt}$ and plug in the known values. Give units.

Problem 2B.

Step 1.

Find an expression for $\frac{dV}{dt}$, where

$$\frac{dV}{dt} = \text{rate in} - \text{rate out.}$$

Step 2.

We have now two expressions for $\frac{dV}{dt}$; the one above and the one in step 4, part 2A. Use that fact to solve for $\frac{dh}{dt}$.

Step 3.

Determine for which values of h we have $\frac{dh}{dt} > 0$ and $\frac{dh}{dt} = 0$. In other words, for which values of h is the depth of the water in the tank increasing. Use that fact to determine if the tank ever can overflow.

4. SOLUTIONS

Problem 1A.

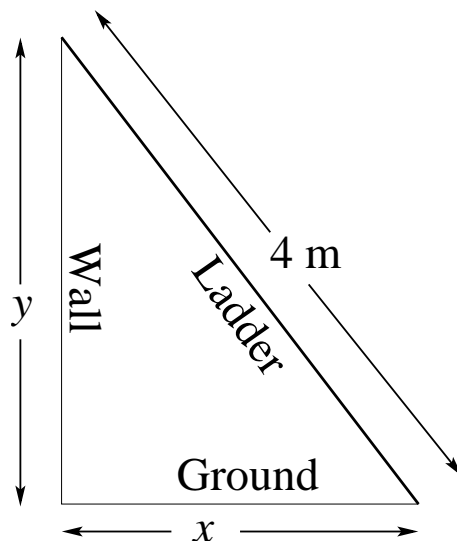


FIGURE 3. The set-up for Problem 1

We have defined the distance of the foot of the ladder from the wall to be x meters and the distance of the top of the ladder from the ground to be y meters. Since these distances are changing with time, we think of x and y as functions of time, t . The rate at which the top of the ladder is falling is given by $\frac{dy}{dt}$ and the rate at which the foot of the ladder moves outward is given by $\frac{dx}{dt}$. We want to determine $\frac{dy}{dt}$ in the moment at which $\frac{dx}{dt} = 0.1$ and $x = 2$. We will need to find a relation between x and y . Since the ladder is 4 meters long, by the Pythagorean Theorem,

$$x^2 + y^2 = 4^2 = 16. \quad (4.1)$$

This equation does not contain neither $\frac{dx}{dt}$ nor $\frac{dy}{dt}$, so the trick here is to find a way to bring in those quantities. We differentiate both sides of the equation in (4.1) with respect to t . By implicit differentiation, remembering that both x and y are functions of t , we have,

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.$$

(On the left side of the equation we used the chain rule, and on the right side we used the fact that the derivative of a constant is zero.) Now, we have an equation that involves both the known quantity $\frac{dx}{dt}$ and the unknown $\frac{dy}{dt}$. Solving for $\frac{dy}{dt}$, we obtain,

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}. \quad (4.2)$$

We need to know what y is when $x = 2$. Since $x^2 + y^2 = 16$, we have

$$y^2 = 16 - x^2 = 16 - 2^2 = 12.$$

Hence, $y = \sqrt{12}$, so the distance of the top of the ladder from the ground is $\sqrt{12}$ m. Plugging in $x = 2$, $y = \sqrt{12}$ and $\frac{dx}{dt} = 0.1$, we have

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\frac{2}{\sqrt{12}} \cdot 0.1 \text{ m/sec} \approx -0.058 \text{ m/sec}.$$

Thus, the top of the ladder falls at 0.058 m/sec. Notice that since y decreases, $\frac{dy}{dt}$ is negative.

Problem 1B.

In part 1A. we constructed the equation,

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}.$$

Since $\frac{dx}{dt}$ is constant, the magnitude of $\frac{dy}{dt}$ is determined by the ratio $\frac{x}{y}$. Since the foot of the ladder moves out, we have that x increases and y decreases. Hence, the magnitude of $\frac{dy}{dt}$ increases as the foot of the ladder moves out.

Problem 2A.

The water in the tank forms a smaller cone with the same angles as the tank. Denote the depth of the water in the tank by h and the radius of the smaller cone with that height by r . Denote the volume of water by V . Now h , r , and V are all functions of time, t . The rate of change of the volume of water is given by $\frac{dV}{dt}$, and the rate of change of the depth of the water in the tank is given by $\frac{dh}{dt}$. In this problem, we want to know $\frac{dh}{dt}$ in the moment at which $h = 5$ and $\frac{dV}{dt} = 0.1$. We need to find a relationship between $\frac{dV}{dt}$ and $\frac{dh}{dt}$. The volume of water in the tank at time t is given by,

$$V = \frac{1}{3}\pi r^2 h.$$

As we can see, this equation contains both r and h , so we need to eliminate r from the formula. From Figure 2, we can see that $\triangle ABC$ and $\triangle ADE$ are similar triangles. Recall that two triangles are similar if the values of their angles are the same. It follows that

$$\frac{BC}{AB} = \frac{DE}{AD},$$

that is

$$\frac{r}{h} = \frac{4}{20} = \frac{1}{5}.$$

Hence,

$$r = \frac{h}{5}.$$

Now, plug this value for r into the volume formula, and we get

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{5}\right)^2 h = \frac{\pi h^3}{75}.$$

We differentiate both sides of the equation with respect to t . We obtain

$$\frac{dV}{dt} = \frac{3h^2\pi}{75} \frac{dh}{dt} = \frac{\pi h^2}{25} \frac{dh}{dt}.$$

Solving for the unknown $\frac{dh}{dt}$, we obtain

$$\frac{dh}{dt} = \frac{25}{\pi h^2} \frac{dV}{dt}.$$

Plugging in the known values, $\frac{dV}{dt} = 0.1$ and $h = 5$, we have

$$\frac{dh}{dt} = \frac{25 \cdot 0.1}{\pi 5^2} \text{ m/min} = \frac{0.1}{\pi} \text{ m/min} \approx 0.032 \text{ m/min}.$$

So the height of the water is increasing at a rate of 0.032 m/min when the depth of the water is 5 m.

Problem 2B. Since water is flowing into the tank at a rate of $0.1 \text{ m}^3/\text{min}$ and leaking out at $2.5 \times 10^{-4} h^2 \text{ m}^3/\text{min}$, we have

$$\frac{dV}{dt} = 0.1 - 2.5 \times 10^{-4} h^2.$$

From part 2A, we also have

$$\frac{dV}{dt} = \frac{\pi h^2}{25} \frac{dh}{dt}.$$

Hence

$$\frac{\pi h^2}{25} \frac{dh}{dt} = 0.1 - 2.5 \times 10^{-4} h^2.$$

Solving for $\frac{dh}{dt}$, we obtain

$$\frac{dh}{dt} = \frac{25 \cdot 2.5 \times 10^{-4} [400 - h^2]}{\pi h^2}.$$

This implies that if $h < 20$, then $\frac{dh}{dt} > 0$. Thus, the depth increases until $h = 20$ m when the tank is full. When $h = 20$, we have $\frac{dh}{dt} = 0$, so the water level does not rise at that point, and hence the tank does not overflow.