Chapter 9

Confidence Intervals

In this chapter, we will discuss the following topics:

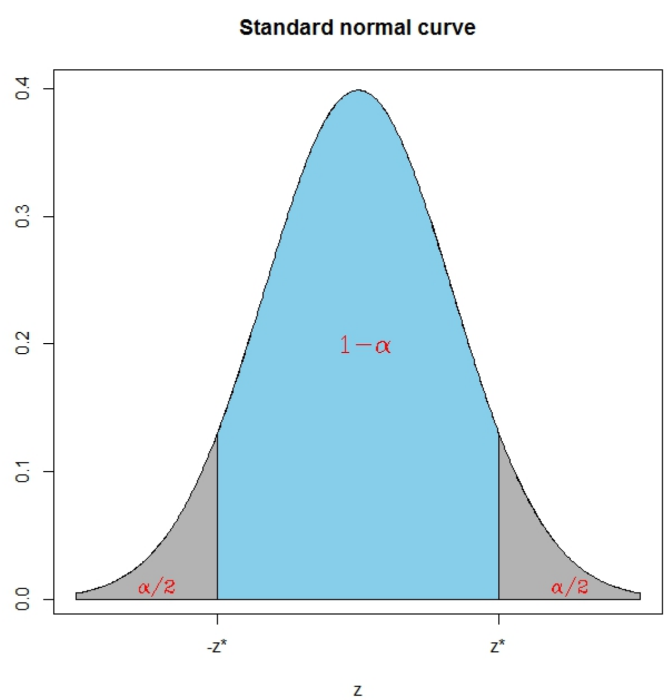
- How to find critical values from the Normal distribution using the R-function **qnorm**.
- How to compute confidence intervals for the mean when the standard deviation is known.

Confidence Interval for a Population Mean with known Standard Deviation

Assume a simple random sample, X_1, X_2..., X_n, of size n is drawn from a Normal population with unknown mean μ and known standard deviation σ. Assume x_1, x_2,..., x_n are the observed values of the sample with mean $\bar{x} = \sum_{i=1}^{n} x_i$. A $100(1-\alpha)\%$ *Confidence interval* for μ is

$$\bar{x} \pm z^* \frac{\sigma}{\sqrt{n}},$$

where the critical value z^* is defined such that the area under the standard normal curve is $1-\alpha$ between $-z^*$ and z^*.
It follows from this definition that for a $100(1-\alpha)\%$ confidence interval, we have an area of $\frac{\alpha}{2}$ both above the critical value z^* and below the critical value $-z^*$. (See the graph below). The sample mean $\overline{X}$ is an unbiased estimator of the population mean μ and we recall that



$\frac{\sigma}{\sqrt{n}}$ is the standard deviation of $\overline{X}$. A $100(1-\alpha)\%$ confidence interval tells us that we are $100(1-\alpha)\%$ certain that the true value of the population mean μ is within the interval in repeated samples. Why does the $100(1-\alpha)\%$ confidence interval look this way? To answer this question, we first standardize the sample mean $\overline{X}$ such that

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

is standard normal. Then we find the critical value z^* such that the area under the standard normal curve between $-z^*$ and z^* is $(1-\alpha)$, that is,

$$1 - \alpha = P(-z^* \leq Z \leq z^*) = P\left(-z^* \leq \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq z^* \right).$$

Now, this expression is equivalent to

$$1 - \alpha = P\left(\overline{X} - z^* \frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + z^* \frac{\sigma}{\sqrt{n}} \right).$$

Then if x_1, x_2,..., x_n are the observed values,

$$\left(\bar{x} - z^* \frac{\sigma}{\sqrt{n}},\ \ \bar{x} + z^* \frac{\sigma}{\sqrt{n}} \right)$$

is a $100(1-\alpha)\%$ confidence interval for the population mean μ.

Problem. Use R to find the critical value z^* for confidence level:
(a) 95%
(b) 80%

Solution to part (a). Here $\alpha = 0.05$ and the total area below the critical value z^* is $1 - \frac{\alpha}{2} = 1 - \frac{0.05}{2} = 0.975$. Also consider the first graph below. If we add together the probabilities in the yellow and blue shade of the graph, we obtain $0.25 + 0.95 = 0.975$. We want to find z^* such that there is a probability of 0.975 that the standard normal random variable, Z, is less than or equal to z^*; that is, $P(Z \leq z^*) = 0.975$. In R this can be done as:

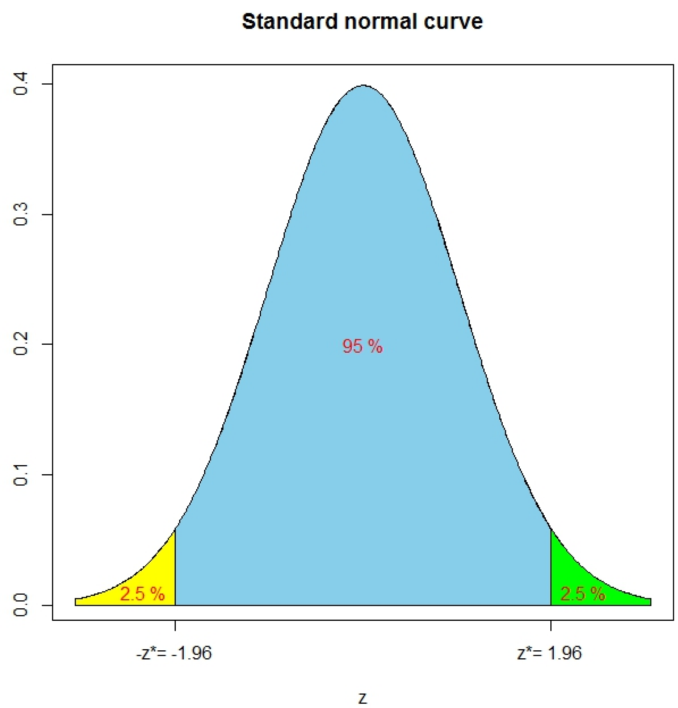
```
> qnorm(0.975)
[1] 1.959964
```

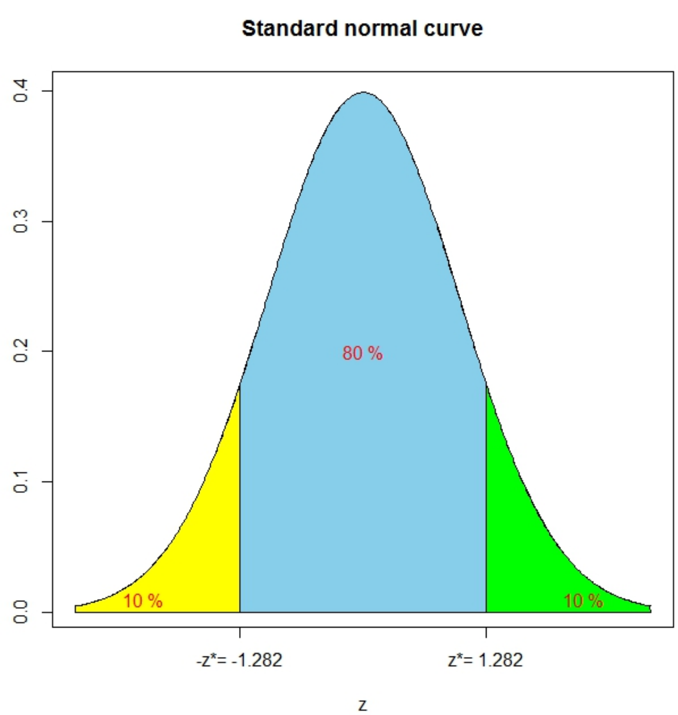
Hence, $z^* = 1.960$.

Solution to part (b). Consider the second graph below. If we add together the probabilities in the yellow and blue shade of the graph, we obtain $0.10 + 0.80 = 0.90$. We want to find z^* such that $P(Z \leq z^*) = 0.90$. In R this can be done as:

```
> qnorm(0.90)
[1] 1.281552
```

Hence, $z^* = 1.282$.





Problem. Suppose that we have a random sample of 25 IQ scores of eight-graders in a city. Suppose that the distribution of IQ scores among all eight-graders in that city is expected to be Normal with unknown mean μ and known standard deviation of $\sigma = 14$. Here are the scores:

```
 107 110  99 131 123  83 143 129 102  72  97 100  92
 118 103 110  90 132 110 139  93 101 102 107  96
```

Verify that the 25 scores do not depart too much from Normality by drawing a stemplot. Then create a 99% confidence interval for the mean IQ score of all eight-graders in the city.

Solution. In R:

```
> IQ=c(107,110,99,131,123,83,143,129,102,72,97,100,92,118,103,110,90,132,
+ 110,139,93,101,102,107,96)
> stem(IQ)

  The decimal point is 1 digit(s) to the right of the |

   6 | 2
   8 | 3023679
  10 | 01223770008
  12 | 39129
  14 | 3
```

We have $1 - \frac{\alpha}{2} = 1 - \frac{0.01}{2} = 1 - 0.005 = 0.995$, so we first want to find the critical value z^* such that $P(Z \leq z^*) = 0.995$. Then we will find a 99% confidence interval. In R:

```
> IQ=c(107,110,99,131,123,83,143,129,102,72,97,100,92,118,103,110,90,132,
+ 110,139,93,101,102,107,96)
> z=qnorm(0.995)
> z
[1] 2.575829
> xbar=mean(IQ)
> xbar
[1] 107.56
> sdx =(14/sqrt(25))
> c(xbar-z*sdx,xbar+z*sdx)
[1] 100.3477 114.7723
```

We obtain from R that $z^* = 2.576$, $\bar{x} = 107.56$, and the 99% confidence interval is $(100.35,\ \ 114.77)$. Thus, we are 99% confident that the mean IQ of eight-graders in the city is between 100.35 and 114.77 points.

Problem. (For more advanced R-users). Create a function for computing the confidence interval in the previous problem:

Solution. We will create a function that calculates the confidence interval for varying values of the standard deviation, confidence level, and sample size:

```
> IQ=c(107,110,99,131,123,83,143,129,102,72,97,100,92,118,103,110,90,132,
+ 110,139,93,101,102,107,96)
```

```
> f=function(IQ,sigma,level,n){z=qnorm(level)
+ xbar=mean(IQ)
+ sdx =(14/sqrt(n))
+ c(xbar-z*sdx,xbar+z*sdx)}
> f(IQ,14,0.995,25)
[1] 100.3477 114.7723
```

Explanation. The code can be explained as follows:

- We create the function by the command **function()**{} and we add the variables *IQ, sigma, level, n* to the function which we named f.
- We call the function f with the following values of the variables: IQ (is already defined), sigma=14, level=0.995, and n=25, in **f(IQ, 14, 0.995, 25)**.

The Margin of Error

The margin of error for a confidence interval is defined as:

$$\text{margin of error} = \text{critical value} \times \text{standard error} = z^* \frac{\sigma}{\sqrt{n}}.$$

Thus, a confidence interval is on the form

$$\text{estimate} \ \pm \ \text{margin of error}.$$

The margin of error decreases if:

- the standard deviation decreases. A low variation in the population reduces the standard error of the estimate.
- the sample size n increases. Increasing the sample size makes the estimate more precise.
- the confidence level decreases. Lowering the confidence level decreases the value of the critical value z^*. Thus, we have to accept lower confidence if we would want to have a lower margin of error.

Problem. Suppose that we have a simple random sample of size 10 from a large population with unknown mean and known standard deviation $\sigma = 9$. Compute the margins of error for 95% confidence. Repeat the problem for sample sizes 40 and 160. What do you notice?

Solution. The margin of error is $z^* \frac{\sigma}{\sqrt{n}} = 1.96 \frac{9}{\sqrt{n}}$. The critical value is found in the first problem in this chapter.
For $n = 10$, the margin of error is 5.58.
For $n = 40$, the margin of error is 2.79.
For $n = 160$, the margin of error is 1.39.
Increasing the sample size by 4 reduces the margin of error to half of the value.

Note that usually the standard deviation σ is unknown. We will consider that situation in a later chapter.