Random Variables and Their Probability Distributions, chapter 4.1

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Random Variables

Numerical outcomes such as

- the number of students who received A in a course
- the number of people who live past the age of 100
- number of accidents at a particular street

whose values can change from experiment to experiment are called random variables.

Definition

A random variable is a real valued function whose domain is the sample space:

$$X : S \to \mathbb{R}.$$

We use capital letters near the end of the alphabet, e.g. X,Y,Z for random variables.

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- There are $2^3 = 8$ possible outcomes.
- Let X be the number of heads observed. X is a random variable and can take the values 0,1,2 or 3.
- Since each outcomes is equally likely, the probability that 0 head is observed is $P(X=0) = \frac{\binom{3}{0}}{8} = \frac{1}{8}$.
  The probability that exactly 1 head is observed is $P(X=1) = \frac{\binom{3}{1}}{8} = \frac{3}{8}$.
  The probability that exactly 2 heads are observed is $P(X=2) = \frac{\binom{3}{2}}{8} = \frac{3}{8}$.

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- The probability that at most 1 head is observed is $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.

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- The probability that at most 1 head is observed is $P(X\leq 1)=P(X=0)+P(X=1)=\frac{1}{8}+\frac{3}{8}=\frac{1}{2}$.
- The probability that more than 1 head is observed is $P(X>1)=P(X=2)+P(X=3)=\frac{3}{8}+\frac{1}{8}=\frac{1}{2}$.

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- The probability that at most 1 head is observed is $P(X \leq 1) = P(X=0) + P(X=1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.
- The probability that more than 1 head is observed is $P(X > 1) = P(X=2) + P(X=3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.
- The probability that more than 1 head is observed can also be found as $P(X > 1) = 1 - P(X \leq 1) = 1 - \frac{1}{2} = \frac{1}{2}$.

Random Variables

- The values that random variables can assume are denoted by lower case letters, such as x,y,z.
- We write $P(X = x) = p(x)$ for the probability that X takes on the value x.
- In our previous coin tossing example, we have

x	$p(x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$
Total	1

What does this notation mean?

Let A be some subset of the range of a discrete random variable. For example $A = \{\text{is greater than } 2\}$, $A = \{\text{between } 0 \text{ and } 2\}$. We write

$$\{e \in S | X(e) \in A\} \tag{1}$$

for the set of outcomes e in S such that, $X(e)$, the value of the random variable is in the subset A. We typically write (1) as $(X \in A)$. For example,
$\{X \text{ is greater than } 2\} = (X > 2) = \{e \in S | X(e) > 2\}$
$\{X \text{ is between } 0 \text{ and } 2\} = (0 < X < 2) = \{e \in S | 0 < X(e) < 2\}$.

Discrete Random Variables

Definition

A random variable $X : S \rightarrow \mathbb{R}$ is said to be discrete if its range,

$$\{X(e) | e \in S\},$$

is either finite or infinite countable, i.e., X can take on only a finite number- or a countable infinite number- of possible values x.

Probability Function

Definition

Let $X(S) \subset \mathbb{R}$ be a random variable. The *Probability function* of X, $\quad p : X(S) \to [0, 1]$ is defined as

$$p(x) = P(\{e \in S | X(e) = x\})$$

for any $x \in X(S)$. Thus, $p(x)$ is the probability that X equals x, and we write

$$p(x) = P(X = x).$$

The probability function, $p(x)$, assigns probability to each value x of X so that the following conditions are satisfied:

- $P(X = x) = p(x) \geq 0$ for all $x \in X(S)$
- $\sum_{x \in X(S)} P(X = x) = 1$, where the sum is over all possible values of x.

Cumulative Probability Distribution (CDF)

Definition

The *(cumulative) distribution function*, $F(b)$, for a random variable X evaluated at b is defined by

$$F(b) = P(X \leq b).$$

If X is discrete,

$$F(b) = \sum_{x=-\infty}^{b} p(x),$$

where $p(x)$ is the probability function.

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-

$$\begin{aligned} P(a < X \leq b) &= P((X \leq b) \setminus (X \leq a)) \\ &= P(X \leq b) - P(X \leq a) = F(b) - F(a). \end{aligned}$$

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- $\lim_{x \to \infty} F(x) = 1$
- F is nondecreasing; that is if $a < b$, $F(a) \leq F(b)$.
- F is right-hand continuous; that is

$$\lim_{x \to x_0^+} F(x) = F(x_0).$$

Example

Suppose a section of an electrical circuit has two relays, I and II, operating in parallel. The current will flow when a switch is thrown if either one or both of the relays close. The probability that a relay will close properly is 0.9. We assume that the relays operate independently. Let E_i denote the event that relay i closes properly when the switch is thrown. Then $P(E_i) = 0.9$. Let X be the number of relays that close properly. X can take the values $0, 1, 2$.

- The probability that 0 relays close is

$$\begin{aligned} p(0) &= P(X = 0) \\ &= P(\bar{E}_1 \cap \bar{E}_2) \\ &= P(\bar{E}_1)P(\bar{E}_2) \\ &= (0.1)(0.1) \\ &= 0.01. \end{aligned}$$

Example continue

- The probability that 1 relays close is

$$
\begin{aligned}
p(1) &= P(X = 1) \\
&= P(E_1 \bar{E}_2 \cup \bar{E}_1 E_2) \\
&= P(E_1 \bar{E}_2) + P(\bar{E}_1 E_2) \\
&= P(E_1)P(\bar{E}_2) + P(\bar{E}_1)P(E_2) \\
&= (0.9)(0.1) + (0.1)(0.9) = 0.18
\end{aligned}
$$

- The probability that 2 relays close is

$$
\begin{aligned}
p(2) &= P(X = 2) \\
&= P(E_1 \cap E_2) \\
&= P(E_1)P(E_2) \\
&= (0.9)(0.9) = 0.81.
\end{aligned}
$$

Example continue

The current will flow if X is equal to at least 1; that is $X \geq 1$. This event has probability,

$$\begin{aligned} P(X \geq 1) &= P((X = 1) \cup (X = 2)) \\ &= P(X = 1) + P(X = 2) \\ &= 0.18 + 0.81 \\ &= 0.99. \end{aligned}$$

Example continue

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- Notice that $F(k) = P(X \leq k) = P(X \leq 2) = F(2)$ for $k \geq 2$.

Example continue

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- $F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.01 + 0.18 + 0.81 = 1$
- Notice that $F(k) = P(X \leq k) = P(X \leq 2) = F(2)$ for $k \geq 2$.
- Distribution Function:

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.01, & 0 \leq x < 1 \\ 0.19, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

Example continue

The height of the step is the probability associated with the value of x. Notice that we have

- $p(1) = P(0 < X \leq 1) = F(1) - F(0) = 0.19 - 0.01 = 0.18$
- $p(2) = P(1 < X \leq 2) = F(2) - F(1) = 1 - 0.19 = 0.81$
- Notice that $\lim_{x \to 1^+} F(x) = 0.19 = F(1)$. So F is right-hand continuous.
- $\lim_{x \to 1^-} F(x) = 0.01 \neq 0.19 = F(1)$. So F is not left-hand continuous.

Example

The distribution function of X, which is the number of times per week a student at a large university exercises, is given as follows,

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.30, & 0 \leq x < 1 \\ 0.50, & 1 \leq x < 2 \\ 0.80, & 2 \leq x < 3 \\ 0.90, & 3 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

Verify that F is a distribution function and find the probability function associated with F.

Solution

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x	$p(x)$
0	$0.30 - 0 = 0.30$
1	$0.50 - 0.30 = 0.20$
2	$0.80 - 0.50 = 0.30$
3	$0.90 - 0.80 = 0.10$
4	$1 - 0.90 = 0.10$

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- The cumulative distribution function is:

$$\begin{aligned} F(x) &= p(0) + p(1) + \ldots + p(x) \\ &= p + (1-p)p + (1-p)^2 p + \ldots + (1-p)^x p \\ &= p\left(\frac{1-(1-p)^{x+1}}{p}\right) = 1-(1-p)^{x+1}. \end{aligned}$$

Example

Suppose that the probability of winning is $p = 0.10$ in the previous example.

(A) What is the probability that one has to play at most 3 unsuccessful games before the first win?

(B) What is the probability that one has to play at least 4 unsuccessful games before the first win?

(C) How many games do you need to play in order for the probability of winning at least once is 0.95 or greater?

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Solution

(A) $F(3) = P(X \leq 3) = 1 - (0.90)^{3+1} = 0.3439$.
(B) $P(X > 3) = 1 - F(3) = 0.90^4 = 0.6561$.

Solutions continue

- In part (B), we can see that we have a 65.61% probability of losing at least 4 times before the first win. In other words, if you play four games in a row, the probability of losing all 4 times is 0.6561 and the probability of winning at least once in the 4 plays is $F(3) = 0.3439$.

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- Thus, in general, $F(x)$ is the probability that you win at least once in $x + 1$ successive plays.

Solutions continue

Solution

(C) $F(x-1) = 1-(0.90)^x$ *is the probability that you win at least once in* x *successive plays. Now let*

$$1-(0.9)^x = 0.95.$$

Then

$$\ln(0.90)^x = \ln(0.05)$$

and hence

$$x = \frac{\ln(0.05)}{\ln(0.90)} \approx 28.43.$$

Thus, you need to play 29 times for the probability to be at least 95% that you win at least once.

R-code

- In the previous example, the terms for the probability mass function, $p(x) = p(X = x) = (1 - p)^x p$, formed a geometric sequence. X is therefore called a *geometric random variable*.
- In R, we can calculate the probability mass function and the cumulative distributon function for the geometric random variable with $p = 0.1$ as follows:

  $> x < -c(0:5)$

  $> f < -dgeom(x, 1/10)$ # d indicates density

  $> F < -pgeom(x, 1/10)$ # p indicates probability

  $> data.frame(x, f, F)$

  $> y < -rgeom(100, 1/10)$ # simulate 100 geometric random variables

  $> hist(y)$

Example

Five lightbulb, 2 of which are defect are being mixed. Two of the lightbulb are randomly selected. Let X denote the number of defective lightbulb picked.
(A) Determine the probability mass function for X.
(B) Determine the Cumulative Distribution function for X.

Solution

- *The probability of selecting 0 defect lightbulbs is* $p(0) = \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}$

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The probability mass function is given by,

x	$p(x)$
0	$\frac{3}{10}$
1	$\frac{3}{5}$
2	$\frac{1}{10}$

(B) The cumulative distribution function is

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.30, & 0 \leq x < 1 \\ 0.90, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$