Discrete Probability Distributions, chapter 4.3-4.8

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February 27, 2011

The Bernoulli Distribution

- Experiments with two possible outcomes.
- A selected lightbulb is either defective or no defective.
- You either win or loose.
- A cat is either pregnant or not pregnant.

Such experiments are called Bernoulli trials.

The Bernoulli Distribution

- Suppose one outcome of a Bernoulli trial is indentified to be a success and the other a failure. Define the random variable X as,
  - $X = 1$ if the outcome is a success.
  - $X = 0$ if the outcome is a failure.

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- X is said to have a Bernoulli distribution.
- The Bernoulli random variable is a bulding block for other probability distributions such as the binomial distribution.

The Bernoulli Distribution

Theorem

The Bernoulli distribution:

$$p(x) = p^x(1-p)^{1-x}, \;\; x = 0, 1 \;\; \textit{for} \;\; 0 < p < 1.$$

$$E(X) = p \quad \textit{and} \quad V(X) = p(1-p).$$

$E(X) = \sum_x p(x) = 0 \cdot p(0) + 1 \cdot p(1) = 0(1-p) + 1(p) = p.$

$$\begin{aligned} V(X) = E(X^2) - [E(X)]^2 &= \sum_x x^2 p(x) - p^2 \\ &= 0^2(1-p) + 1^2(p) - p^2 \\ &= p - p^2 = p(1-p). \end{aligned}$$

The Binomial Distribution

- Suppose we perform n independent Bernoulli trials.
- Each trials have probability p of success.
- Let the random variable X be the number of successes in the n trials.

The distribution of X is called the Binomial distribution.

The Binomial Distribution

- Suppose we insepct n items independently and record values for X_1, X_2,..., X_n, where

$$X_i = \begin{cases} 1, & \text{if the } i^{th} \text{ item is defective} \\ 0, & \text{if the } i^{th} \text{ item is not defective} \end{cases}$$

- Then

$$X = \sum_{i=1}^{n} X_i$$

denotes the number of defective items among the n items.

- We assume that $P(X_i = 1) = p$ for each i.

The Binomial Distribution

Suppose X_1, X_2, ..., X_n are n independent Bernoulli random variables with

$$X = \sum_{i=1}^{n} X_i.$$

Then

$$E(X) = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np.$$

$$V(X) = \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} p(1-p) = np(1-p).$$

The Binomial Distribution

Theorem

The Binomial Distribution:

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \;\; x = 0, 1, 2, ..., n \;\; for \;\; 0 \leq p \leq 1.$$

$$E(X) = np \;\; and \;\; V(X) = np(1-p).$$

Here the probability that $X = x$ is given by the term $p^x(1-p)^{n-x}$ multiplied by the number of ways of selecting x positions for defectives in the n possible positions. This number is $\binom{n}{x} = \frac{n!}{x!(n-x)!}$.

The Binomial Distribution

Recall the binomial expansion:

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}.$$

Observe, using the binomial expansion with $a = p$, $b = (1-p)$ that

$$\sum_x p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = 1.$$

Summary, Binomial Distribution

A random variable X has a binomial distribution if:

- The experiment consists of n identical trials.
- Each trial has exactly two outcomes; a success or a failure, that is each trial is a Bernoulli trial.
- The probability of success, p, is constant from trial to trial.
- The trials are independent.
- X is the number of successes among the n trials.

Examples.

- Number of defectives in a sample of n items.
- Number of heads in a sequence of n coin tosses.
- Number of people with type O^+ blood of the n people that enter a blood bank.
- Number of children of a couple who has a genetic disease.

R-code

Let n be the number of trials.
Let $X = x$ be the number of successes among the n trials.
Let p be the probability for success.
Then in R,
$> p < -dbinom(x, n, p)$ $\quad$ # probability of x successes among the n trials.
$> F < -pbinom(x, n, p)$ $\quad$ # the probability of at most x successes among the n trials which is F(x)).

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$$P(X \leq 8) = F(8) = \sum_{x=0}^{8} p(x).$$

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- The probability of at least 8 successes is

$$P(X \geq 8) = 1 - F(7) = \sum_{x=8}^{n} p(x).$$

- The probability of more than 3 successes but at most 8 successes is

$$P(3 < X \leq 8) = F(8) - F(3) = \sum_{x=4}^{8} p(x).$$

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- The probability of more than 3 successes but fewer than 8 successes is

$$P(3 < X < 8) = F(7) - F(3) = \sum_{x=4}^{7} p(x).$$

Example

Suppose a student take a final multiple choice exam with 20 questions. The student has not studied during the whole semester, so he plans to guess on each question. Each question has 5 choices.
(A) Determine the probability that the student guesses correctly on 7 of the questions.
(B) Determine the probability that the student guesses correctly on at least one question.

Example

Suppose a student take a final multiple choice exam with 20 questions. The student has not studied during the whole semester, so he plans to guess on each question. Each question has 5 choices.
(A) Determine the probability that the student guesses correctly on 7 of the questions.
(B) Determine the probability that the student guesses correctly on at least one question.

Solution

There are $n = 20$ *identical trials.*
Each trial has two outcomes, right and wrong.
The probability of success is $p = \frac{1}{5}$ *on each trial.*
The guesses are independent.
Let X *be the number of questions the student guesses correctly.*

Solution

(A) The probability that the student guesses correctly on 7 questions is

$$p(7) = \binom{20}{7}(0.20)^7(0.80)^{13} = 0.0545.$$

In R:
$> p < -dbinom(7, 20, 0.2)$
[1]0.0545
(B) The probability that the student guesses correctly on at least one question is

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{20}{0}(0.20)^0(0.80)^{20} = 0.988.$$

Problem 4.55

A firm sells four items randomly selected from a large lot that is known to contain 12% defectives. Let X denote the number of defectives among the four sold. The purchaser of the items will return the defectives for repair, and the repair cost is given by

$$C = 2X^2 + X + 3.$$

Find the expected repair cost.

Problem 4.63

A farmer hires a consultant to tell him whether or not he needs to spray his cotton crop to control insects. In any given year, the probability that treatment is necessary to prevent serious economics loss is 0.6. If treatment is necessary, the consultant recommends treatment 99% of the time. If treatment is not truly necessary, the consultant recommends treatment 40% of the time. The farmer always follows the recommendation.
(A) What is the probability that the farmer will spray 3 years straight?
(B) The cost of spraying is C. Without considering the cost of spraying, the farmer's profit is P_1 if no treatment is needed or if treatment is given (whether or not it is needed) and P_2 if treatment is needed but not given. Find the expected value of profit, taking into consideration the cost of spraying over the 3 years.

The Geometric Distribution

- Suppose a series of trials can be represented by a sequence of independent Bernoulli random variables with $Y_i = 1$ if the i^{th} trial is a success and $Y_i = 0$ otherwise.
- Assume the probability, p of success is constant for each trial.
- We are interested in the number of failures prior to the first success.
- Let X denote the number of failures prior to the first success.

The Geometric Distribution

Then the geometric probability distribution is

$$\begin{aligned} P(X = x) &= p(x) \\ &= P(Y_1 = 0, Y_2 = 0, ..., Y_x = 0, Y_{x+1} = 1) \\ &= P(Y_1 = 0)P(Y_2 = 0) \cdots P(Y_x = 0)P(Y_{x+1} = 1) \\ &= (1 - p)^x p \\ &= q^x p \end{aligned}$$

for $x = 0, 1, 2, ...$

The Geometric Distribution

Notice

$$\begin{aligned} P(X = x) &= q^x p = q[q^{x-1} p] \\ &= qP(X = x - 1) \\ &= (1 - p)P(X = x - 1) \\ &< P(X = x - 1) \end{aligned}$$

for $x = 1, 2, \ldots$ Each succedding probability is less than the previous one.

Examples, The Geometric Distribution

- Number of tails before a head
- Number of times a person is exposed to a disease before contracting the disease.
- Number of unqualified applicants before a qualified applicant.

Geometric Series

Recall the sum of a geometric series:

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}.$$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1 - r} \quad \text{for} \quad |r| < 1.$$

Then

$$\sum_{x} p(x) = \sum_{x=0}^{\infty} (1 - p)^x p = p \sum_{x=0}^{\infty} (1 - p)^x = p \frac{1}{1 - (1 - p)} = 1.$$

The Geometric Distribution

For any integer $x \geq 0$

$$F(x) = P(X \leq x) = \sum_{i=0}^{x} q^i p = p \sum_{i=0}^{x} q^i$$
$$= p\frac{1-q^{x+1}}{1-q} = p\frac{1-q^{x+1}}{p} = 1 - q^{x+1}.$$

For any integer $x \geq 0$,

$$P(X \geq x) = 1 - F(x-1) = 1 - (1 - q^x) = q^x.$$

The Expected value of the Geometric Distribution

$$E(X) = \sum_{x=0}^{\infty} xpq^x = p \sum_{x=0}^{\infty} xq^x$$

$$= p[0 + q + 2q^2 + 3q^3 + \cdots] \quad = pq[1 + 2q + 3q^2 + \cdots]$$

$$= pq[1 + q + q^2 + \cdots$$

$$q + q^2 + \cdots$$

$$q^2 + \cdots]$$

$$= pq[\frac{1}{p} + \frac{q}{p} + \frac{q^2}{p} + \cdots]$$

$$= q[1 + q + q^2 + \cdots]$$

$$= \frac{q}{1-q} = \frac{q}{p}$$

The Geometric Distribution

Theorem

The Geometric Distribution:

$$p(x) = p(1-p)^x, \;\; x = 0, 1, 2, \ldots \;\; \textit{for} \;\; 0 \le p \le 1.$$

$$E(X) = \frac{q}{p} \;\; \textit{for} \;\; V(X) = \frac{q}{p^2}.$$

The Geometric Distribution

The Geometric Distribution has the memoryless property. That is for integers, $j, k > 0$,

$$P(X \geq j + k | X \geq j) = P(X \geq k).$$

Proof.

$$P(X \geq j + k | X \geq j) = \frac{P((X \geq j + k) \cap (X \geq j))}{P(X \geq j)} = \frac{P(X \geq j + k)}{P(X \geq j)}$$

$$= \frac{q^{j+k}}{q^j} = q^k = P(X \geq k).$$

□

The Geometric Distribution

Example

A company finds that if a care salesman visits with a customer the probability that the customer will puchase a car is 0.10. Let X be the number of customers visited before the first sale is made.
(A) Find the probability that 4 customers are visited before the first sale is made.
(B) Suppose it costs $50 for a company for each visit with a customer. Find the expected value and variance of the total cost to visit with customers until the first sale is made.

The Geometric Distribution

Solution

(A)

$$P(X = 4) = p(4) = (0.90)^4(0.10) = 0.066.$$

(B) Since $(X + 1)$ *is the number of trials on which the sales are performed, the total cost of visits with customer is* $C = 50(X + 1)$*. Now*

$$E(C) = 50E(X) + 50 = 50\frac{q}{p} + 50 = 50\frac{0.90}{0.10} + 50 = 500.$$

$$V(C) = (50^2)\frac{q}{p^2} = 2500\frac{0.90}{0.10^2} = 225000.$$

The Negative Binomial Distribution

- Now we are interested in the number of failures prior to the k^{th} success.
- Let X denote the number of failures prior to the k^{th} success in a sequence of independent Bernoulli trials.
- Let p denote the probability of success.

The Negative Binomial Distribution

Now $P(X = x)$ is the probability that the 1^{th} $(x + k - 1)$ trials contain $(k - 1)$ successes and the $(x + k)^{th}$ trial is a success. We can write this as

$$\begin{aligned} P(X = x) &= P(1^{th}(x + k - 1) \text{ trials contain } (k-1) \text{ successes}) \times \\ &\times P((x+k)^{th} \text{ is a success}) \\ &= \binom{x+k-1}{k-1} p^{k-1}(1-p)^x p \\ &= \binom{x+k-1}{k-1} p^k q^x \end{aligned}$$

for $x = 0, 1, \ldots$

The expectation and variance of the Negative Binomial Distribution

Let X denote the number of failures prior to the k^{th} success.
Let X_1 denote the number of failures prior to the first success.
Let X_2 denote the number of failures between the first and the second success.
Let X_3 denote the number of failures between the second and the third success and so forth.
Then we have $X = \sum_{i=1}^{k} X_i$, where each X_i are independent and each has a geometric distribution. Then

$$E(X) = \sum_{i=1}^{k} E(X_i) = \sum_{i=1}^{k} \frac{q}{p} = k\frac{q}{p}.$$

$$V(X) = \sum^{k} V(X_i) = \sum^{k} \frac{q}{p^2} = k\frac{q}{p^2}.$$

The Negative Binomial Distribution

Theorem

The Negative Binomial Distribution:

$$p(x) = \binom{x+k-1}{k-1} p^k (1-p)^x \quad x = 0, 1, 2, \ldots, \quad \text{for } 0 \leq p \leq 1.$$

$$E(X) = \frac{kq}{p} \text{ and } V(X) = \frac{kq}{p^2}.$$

The Negative Binomial Distribution

Example

In the previous example, the probability that a customer purchase a car if a car salesman visits with them is 0.10. Suppose he must sell 3 car each day.

(A) Find the probability that the salesman will have to visit with 7 customers, who are not interested in buying cars, before making 3 sales.

(B) Find the probability that the salesman will have to visit with 8 customers total.

(C) If the cost of visiting with a customer is $50. Determine the expected value and the variance of the number of visits that must be made to sell 3 cars.

(D) Find the probability that the salesman will have to visit with more than 10 customers before making 3 sales.

The Negative Binomial Distribution

Solution

(A)

$$P(X=7)=\binom{7+3-1}{3-1}(0.1)^3(0.9)^7=\binom{9}{2}(0.1)^3(0.9)^7=0.0172.$$

(B)

$$P(X=5)=\binom{5+3-1}{3-1}(0.1)^3(0.9)^5=\binom{7}{2}(0.1)^3(0.9)^5=0.0124.$$

The cost of visiting with customers until 3 sales are made is $C=X+3$. *Then* $E(C)=E(X+3)=E(X)+3=3\frac{0.9}{0.1}+3=30$ *and* $V(C)=V(X+3)=V(X)=3\frac{0.9}{(0.1)^2}=270$.

Solution continue, R-code

(D)

$$P(X > 7) = 1 - P(X \leq 7) = 1 - \sum_{x=0}^{7} \binom{x+3-1}{3-1} (0.1)^3 (0.9)^x$$
$$= 0.9298092$$

In R:

$p <-$ 1-pnbinom(7,3,0.1)

$<$ 0.9298092

In general: dnbinom(events=x,size=k,prob=p) # the negative binomial probability distribution

pnbinom(events=x,size=k,prob=p) # the negative binomial Cumulative distribution

The Poisson Distribution

Theorem

$$p(x) = \frac{\lambda^x}{x!} e^{-\lambda}, \;\; x = 0, 1, 2, ..., \;\; \textit{for} \;\; \lambda > 0.$$

$$E(X) = \lambda \;\; \textit{and} \;\; V(X) = \lambda.$$

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Notice that

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1.$$

The Poisson Distribution

- λ denotes the mean number of occurences in one time period. In n nonoverlapping time periods, the mean is $n\lambda$.
- The Poisson distribution approximates the binomial distribution with probability p and n trials if n is large, p is small and $\lambda = np$, that is

$$\binom{n}{x} p^x (1-p)^{n-x} \approx \frac{\lambda^x}{x!} e^{-\lambda}.$$

The expectations of The Poisson Distribution

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} e^{-\lambda} \\
&= \sum_{x=1}^{\infty} x \frac{\lambda^x}{x!} e^{-\lambda} \\
&= \lambda e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^{x-1}}{(x-1)!} \\
&= \lambda e^{-\lambda} (1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots) \\
&= \lambda e^{-\lambda} e^{\lambda} = \lambda
\end{aligned}
$$

The calculation of the variance is left as homework.

Applications of the Poisson Distribution

- The number of phone calls in a 10 min period.
- The number of accidents at a particular intersection in a period of a year.
- The number of daily repairs that a machine requires.
- The number of salmonella bacterias in a sample of 1 cubic-centimeter of water.

Poisson versus Binomial

Example

A hunter hunts in a forrest containing $n = 10^5$ rabits. The hunter is able to cach on the average $\lambda = 10$ rabits per day. Determine the probability distribution of the number of rabits caught on a randomly selected day. Then determine the probability of catching 10 rabits per day.

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Solution

- *Assume that each rabit is caught independently of each other.*
- *Each catch is a Bernoulli trial.*
- *Assume that the probability, p, of catching a rabit is the same for every rabit.*
- *We assume $\lambda = np$. Hence $p = \frac{\lambda}{n} = \frac{10}{10^5} = 10^{-4}$.*

Binomial Probability distribution

Let X be the number of rabits caught on a randomly chosen day. Then

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{10^5}{k} (0.0001)^k (0.9999)^{10^5 - k}.$$

Inconvenient formula.

Poisson Probability distribution

The probability of catching a rabit is $p = 10^{-4}$ which is relatively small. We have

$$P(X = k) = \binom{10^5}{k} (0.0001)^k (0.9999)^{10^5 - k} \approx \frac{10^k}{k!} e^{-10}.$$

R-calculation, Poisson and Binomial

$n < -10^5$
$\lambda < -10$ # The mean
$p < -\frac{\lambda}{n}$ # probability for success
$k < -10$ # The number of events
$P1 < -dbinom(k, n, p)$ # The Binomial distribution
$P2 < -dpois(k, \lambda)$ # The Poisson distribution
$P1 = 0.1251163$
$P2 = 0.1251100.$
Thus, the probability of catching 10 rabits is 12.5%.

R-calculation, Poisson versus Binomial

$n < -10^4 : 10^5$ # Range of n.
$\lambda < -10$ # The mean
$p - \frac{\lambda}{n}$ # probability for success
$x < -10$ # The number of events
$<$ plot(n,dbinom(x,n,p)/dpois(x,lambda),xlab="x",
ylab="Binomial/Poisson") # Plot the ratio of Binomial by Poisson

Example

The number of phone calls passing through a particular relay system averages 4 per minute.
(A) Find the probability that two calls will pass through the relay system during a 1-minute period.
(B) Find the probability that no calls will pass through the relay system during a 2-minute period.
(C) Find the probability that at least eight calls will pass through the relay system during a 1-minute period.
(D) Suppose a phone conversation takes approximately 5 min. Determine the mean and variance of the total of service time connected to each phone call passing through the relay system during a 1-minute period.

Solution

$(A)\ p(2) = \frac{4^2}{2!}e^{-4} = 0.147.$

In R:

p <-dpois(2,4)

$<$ *0.147*

(B) *The mean number of calls during a 2-minutes period is* 8 *since the mean number of calls during a 1-minute period is 4.*

Then $p(0) = \frac{8^0}{0!}e^{-8} = 0.000335.$

$(C)\ P(X \geq 8) = 1 - P(X \leq 7) = 1 - 0.949 = 0.051.$

In R: $p < -$ *1-ppois(7,4)* # *ppois is the Poisson cumulative distribution.*

$<$ *0.051*

(D) Let S *be the service time. Then* $S = 5X$*. We have*

$E(S) = E(5X) = 5E(X) = 5\lambda = 5 \cdot 4 = 20 min.$

$V(S) = V(5X) = 5^2 V(X) = 25 \cdot 4 = 100 min^2.$