Expected Values of Random Variables, chapter 4.2

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Expected Value

Example

- Suppse you and your friend are matching balanced coins. Each of you flip a fair coin. If the upper faces match, you win $1; if they do not match, you lose $1. The probability of winning a match is $\frac{1}{2}$, so in the long run you should win about half of the time.

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- On average, you will win the following amount per game in the long run, $(-1)\frac{1}{2} + (1)\frac{1}{2} = 0$ which is the expected value of your winnings. This is a fair game.

Expected Value

Definition

The expected value of a discrete random variable with probability function $p : \mathbb{R} \to \mathbb{R}$ is given by

$$E(X) = \sum_{x \in \mathbb{R}} x p(x).$$

We usually denote $E(X) = \mu$.

Note that in order for the expected value of a discrete random variable X to exist, we must have

$$\sum_{x \in \mathbb{R}} |x| p(x) < \infty \quad \text{(converges absolutely)}.$$

Example

- Now suppose you instead win \$5 per game of matching coins. Your expected winnings per game is $(-5)\frac{1}{2} + 5\frac{1}{2} = 0$.
- Now if X represents your winnings per game when you are playing for \$1, then $5X$ represents your winnings per game when you play for \$5.

For a random variable, we can form functions of X, say $g(X)$.

Theorem

If X is a discrete random variable with probability distribution $p(x)$ and if $g(x)$ is any real-valued function of X, then

$$E(g(X)) = \sum_{x \in \mathbb{R}} g(x)p(x).$$

Example

- Now suppose you win \$1 if the match is tails and \$2 if the match is heads. You loose \$1.50 if the coins do not match.
- Then your expected winnings per game is $(-1.5)\frac{1}{2} + (1)\frac{1}{4} + (2)\frac{1}{4} = 0$.

- In the first game, it is more likely for the random variable to take on values closer to the mean than in the second experiment. For example, in the first experiment, the random variable take on one of the values -1 or 1 with probability 1. Thus, it deviates by 1 unit from the mean.
- In the second experiment, the positive values average 1.5 units from the expected value of 0 as do the negative values.

Variance

- If X is a finite discrete random variable with expected value μ and X takes on the value x when the experiment is performed, then $x - \mu$ is called the deviation of x from μ.
- For example a deviation of $+1$ means x is 1 unit to the right of μ and -1 means 1 unit to the left of μ. In order to not worry about the sign difference, we work with the square of the deviation, $(x - \mu)^2$, instead.

Definition

The variance of a random variable X with expected value μ is given by

$$V(X) = E[(X - \mu)^2].$$

We usually denote $V(x) = \sigma^2$.

Standard deviation

The units associated with σ^2 are the square of the units of measurement for X. The smallest value for σ^2 is zero, which happens when X takes on a constant value with probability 1. As the points with positive probability spread out, the variance becomes larger.

Definition

The standard deviation of a random variable X is

$$\sigma = \sqrt{\sigma^2} = \sqrt{E[(x-\mu)^2]}.$$

The standard deviation maintains the original units of measure.

Variance and Standard Deviation

Example

For the first coin experiment with $\mu = 0$, we have

- $\sigma^2 = E[(x-\mu)^2] = (-1)^2\frac{1}{2} + (1)^2\frac{1}{2} = 1$
- $\sigma = 1.$

For the second game,

- $\sigma^2 = (-1.5)^2\frac{1}{2} + (1)^2\frac{1}{4} + 2^2\frac{1}{4} = 2.375$
- $\sigma = \sqrt{2.375} = 1.54.$

Example

A company is calculating the cost of equipment maintenance for its budget. In order to decide the budget, the number of machine malfunctions per month is considered. Based on past performance, the distribution of X is given as follows:

x	$p(x)$
0	0.1
1	0.3
2	0.4
3	0.2

Determine the expected value, variance and standard deviation of the number of malfunctions per month.

Solution

The expected number of malfunctions per month is

$$E(x) = \sum_x xp(x) = 0(0.1) + 1(0.3) + 2(0.4) + 3(0.2) = 1.7$$

The variance is

$$\begin{aligned} V(x) &= \sum_x (x - 1.7)^2 p(x) \\ &= (0 - 1.7)^2(0.1) + (1 - 1.7)^2(0.3) + \\ &+ (2 - 1.7)^2(0.4) + (3 - 1.7)^2(0.2) \\ &= 0.81 \;\; \textit{times squared per month.} \end{aligned}$$

The standard deviation is

$$\sigma = \sqrt{0.81} = 0.9 \;\; \textit{times per month.}$$

R-commands

$> x < -c(0:3)$
$> p < -c(0.1, 0.3, 0.4, 0.2)$
$> E < -sum(x * p)$ # Expectations
$> V < -sum((x - E)^2 * p)$ # Variance
$> s < -sqrt(V)$ # Standard deviation
$> E$
$> V$
$> s$

Theorem

For any random variable X and constants a and b,
1. $E(aX + b) = aE(X) + b$
2. $V(aX + b) = a^2 V(X)$

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Theorem

If X is a random variable with mean μ, then

$$V(X) = E(X^2) - \mu^2.$$

Summary

Suppse X and Y are random variables defined on a sample space and a is a constant. Then

- $E(X+Y) = E(X) + E(Y)$
- $E(aX) = aE(X)$
- $E(a) = a$
- $E(aX+b) = aE(X) + b$
- $V(X+b) = V(X)$
- $V(aX) = a^2V(X)$
- $V(aX+b) = a^2V(X)$

  If X and Y are independent, then

  $$V(X+Y) = V(X) + V(Y) \text{ (More on this in chapter 6).}$$

Example

The company in the previous example wants to consider the cost of maintenance. It costs \$100 to pay for a representative to check out malfunctions, and the parts cost an average \$80 to correct each malfunction. Find the expected value, variance and mean of the monthly cost of visits by the representative.

Solution

The monthly cost is

$$80X + 100,$$

where X *is the number of malfunctions per month. Then the expected cost per month is*

$$E(80X + 100) = 80E(X) + 100 = 80(1.7) + 100 = 236 \ \ \textit{dollar.}$$

Solution continue

The variance is

$$V(80X+100) = 80^2 V(X) = 80^2(0.81) = 5184 \text{ dollar}^2 \text{ per month.}$$

The standard deviation is

$$\sigma = \sqrt{5184} = 72 \text{ dollar per month.}$$

Coin tossing example

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Suppose a fair coin is tossed n times and let X be the number of heads.

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- The expected value of X is

$$E(X) = \sum_{k=0}^{n} k \frac{1}{2^n}\binom{n}{k} = \frac{n}{2}. \tag{1}$$

- The variance of X is

$$V(X) = \frac{n}{4}. \tag{2}$$

Expected value of number of heads in coin tossing example

Proof.

We will first prove 1 by two methods.
1. method. We will need the identities:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \text{ and } k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Then

$$E(X) = \sum_{k=0}^{n} k \frac{1}{2^n} \binom{n}{k} = \frac{n}{2^n} \sum_{k=0}^{n} \binom{n-1}{k-1} = \frac{n}{2^n} 2^{n-1} = \frac{n}{2}.$$

□

Expected value of number of heads in coin tossing example

Proof.

2. method.

We can write the number of heads in the n coin tosses as

$$X = \sum_{k=1}^{n} X_k,$$

where X_k is the random variable

$$X_k = \begin{cases} 1, & \text{if head on the } k^{th} \text{ toss} \\ 0, & \text{if tail on the } k^{th} \text{ toss} \end{cases}$$

Then $E(X_k) = \frac{1}{2}$ and $E(X) = \sum_{k=1}^{n} E(X_k) = \frac{n}{2}$. □

Variance of number of heads in coin tossing example

The variance for a single coin toss is

$$V(X) = \left(0 - \frac{1}{2}\right)^2 \frac{1}{2} + \left(1 - \frac{1}{2}\right)^2 \frac{1}{2} = \frac{1}{4}.$$

Since X_1, X_2,..., X_n is a sequence of independent random variables, we have

$$V(X) = \sum_{k=0}^{n} V(X_k) = \frac{n}{4}.$$

Infinite expected value

Let X be a random variable with values $k = 1, 2, 3, 4, \ldots$ and with probability mass function,

$$p(k) = \frac{8}{\pi^2}\left(\frac{1}{2k-1}\right)^2.$$

It can be checked that $\sum_k p(k) = 1$. Then

$$E(X) = \sum_{k=1}^{\infty} kp(k) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{k}{(2k-1)^2} \geq \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{k}{4k^2} = \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k} = +\infty$$

Hence X is a random variable with an infinite expected value.

"Standardized" form of X

Suppose $E(X) = \mu$ and $V(X) = \sigma^2$ for a random variable X.
Define the "standardized" form of X, as

$$Y = \frac{X - \mu}{\sigma}.$$

Then $E(Y) = 0$ and $V(Y) = 1$.

Tchebysheff's Theorem

Theorem

Tchebysheff's Theorem: Let X be a random variable with mean μ and variance σ^2. Then for any positive k,

$$P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

The inequality in the statement is equivalent to

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}.$$

If for example $k = 2$, the interval $\mu - 2\sigma < X < \mu + 2\sigma$ contains at least $1 - \frac{1}{2^2} = \frac{3}{4}$ of the random variable X's probability.

Example

Suppose a certain light bulb has an expected lifetime of $\mu = 500$ hours with a variance $\sigma^2 = 100$.
(A) Estimate the probability that one of these light bulb will live between 485 and 515 hours.
(B) Find the shortest interval certain to contain at least 95% of the lifetimes of the light bulbs.

Solution

- *The standard deviation is* $\sigma = 10$. *The interval from 485 to 515 represents* $k = \frac{15}{\sigma} = \frac{15}{10} = 1.5$ *units.*

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- *Thus, the interval from 485 to 515 represents* $500 - \frac{3}{2} \cdot 10$ *to* $500 + \frac{3}{2} \cdot 10$.

Solution

- *The standard deviation is* $\sigma = 10$. *The interval from 485 to 515 represents* $k = \frac{15}{\sigma} = \frac{15}{10} = 1.5$ *units.*
- *Thus, the interval from 485 to 515 represents* $500 - \frac{3}{2} \cdot 10$ *to* $500 + \frac{3}{2} \cdot 10$.
- *We want the probability that the lifetime is within* $k = \frac{3}{2}\ \sigma$ *units of the mean.*

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- *Thus, the interval from 485 to 515 represents* $500 - \frac{3}{2} \cdot 10$ *to* $500 + \frac{3}{2} \cdot 10$.
- *We want the probability that the lifetime is within* $k = \frac{3}{2}\ \sigma$ *units of the mean.*
- *Now* $1 - \frac{1}{k^2} = 1 - \frac{1}{1.5^2} = \frac{5}{9} \approx 0.556$.

Solution

- *The standard deviation is* $\sigma = 10$. *The interval from 485 to 515 represents* $k = \frac{15}{\sigma} = \frac{15}{10} = 1.5$ *units.*
- *Thus, the interval from 485 to 515 represents* $500 - \frac{3}{2} \cdot 10$ *to* $500 + \frac{3}{2} \cdot 10$.
- *We want the probability that the lifetime is within* $k = \frac{3}{2}\ \sigma$ *units of the mean.*
- *Now* $1 - \frac{1}{k^2} = 1 - \frac{1}{1.5^2} = \frac{5}{9} \approx 0.556$.
- *By Tchebysheff's Theorem,* $P(|X - 500| < \frac{3}{2}\sigma) \geq \frac{5}{9} \approx 0.556$. *Thus the probability that the lifetme of a light bulb is within* $\frac{3}{2}\sigma$ *units of the mean is at least* 55.6%.

Solution

- *We must have* $1 - \frac{1}{k^2} = 0.95$*. Hence* $k = \sqrt{20} \approx 4.47$*.*

Solution

- *We must have* $1 - \frac{1}{k^2} = 0.95$. *Hence* $k = \sqrt{20} \approx 4.47$.
- *The interval* $(\mu - 4.47\sigma, \mu + 4.47\sigma)$ *or* $(500 - 4.47(10), 500 + 4.47(10))$ *or* $(455.3, 544.7)$ *will contain at least* 95% *of the lifetimes of the light bulb.*