Random Variables and Their Probability Distributions, chapter 4.1

Grethe Hystad

September 18, 2013

Random Variables

Numerical outcomes such as

- the number of students who received A in a course
- the number of people who live past the age of 100
- number of accidents at a particular street

whose values can change from experiment to experiment are called random variables.

Definition

A random variable is a real valued function whose domain is the sample space:

$$X : S \to \mathbb{R}.$$

We use capital letters near the end of the alphabet, e.g. X,Y,Z for random variables.

Random Variables

Example

- Suppose a fair coin is tossed 3 times.

Random Variables

Example

- Suppose a fair coin is tossed 3 times.
- There are $2^3 = 8$ possible outcomes.

Random Variables

Example

- Suppose a fair coin is tossed 3 times.
- There are $2^3 = 8$ possible outcomes.
- Let X be the number of heads observed. X is a random variable and can take the values 0,1,2 or 3.

Random Variables

Example

- Suppose a fair coin is tossed 3 times.
- There are $2^3 = 8$ possible outcomes.
- Let X be the number of heads observed. X is a random variable and can take the values 0,1,2 or 3.
- Since each outcomes is equally likely, the probability that 0 head is observed is $P(X=0) = \frac{\binom{3}{0}}{8} = \frac{1}{8}$.
  The probability that exactly 1 head is observed is $P(X=1) = \frac{\binom{3}{1}}{8} = \frac{3}{8}$.
  The probability that exactly 2 heads are observed is $P(X=2) = \frac{\binom{3}{2}}{8} = \frac{3}{8}$.

Example

- The probability that exactly 3 heads are observed is $P(X = 3) = \frac{\binom{3}{3}}{8} = \frac{1}{8}$.

Random Variables

Example

- The probability that exactly 3 heads are observed is $P(X = 3) = \frac{\binom{3}{3}}{8} = \frac{1}{8}$.
- The probability that at least 2 heads are observed is $P(X \geq 2) = P(X = 2 \quad \text{or} \quad X = 3) = P(X = 2) + P(X = 3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.

Random Variables

Example

- The probability that exactly 3 heads are observed is $P(X = 3) = \frac{\binom{3}{3}}{8} = \frac{1}{8}$.
- The probability that at least 2 heads are observed is $P(X \geq 2) = P(X = 2 \quad \text{or} \quad X = 3) = P(X = 2) + P(X = 3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.
- The probability that at most 1 head is observed is $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.

Random Variables

Example

- The probability that exactly 3 heads are observed is $P(X = 3) = \frac{\binom{3}{3}}{8} = \frac{1}{8}$.
- The probability that at least 2 heads are observed is $P(X \geq 2) = P(X = 2 \quad \text{or} \quad X = 3) =$ $P(X = 2) + P(X = 3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.
- The probability that at most 1 head is observed is $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.
- The probability that more than 1 head is observed is $P(X > 1) = P(X = 2) + P(X = 3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.

Random Variables

Example

- The probability that exactly 3 heads are observed is $P(X=3) = \frac{\binom{3}{3}}{8} = \frac{1}{8}$.
- The probability that at least 2 heads are observed is $P(X \geq 2) = P(X = 2 \quad \text{or} \quad X = 3) = P(X=2) + P(X=3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.
- The probability that at most 1 head is observed is $P(X \leq 1) = P(X=0) + P(X=1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.
- The probability that more than 1 head is observed is $P(X > 1) = P(X=2) + P(X=3) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.
- The probability that more than 1 head is observed can also be found as $P(X > 1) = 1 - P(X \leq 1) = 1 - \frac{1}{2} = \frac{1}{2}$.

Random Variables

- The values that random variables can assume are denoted by lower case letters, such as x,y,z.
- We write $P(X = x) = p(x)$ for the probability that X takes on the value x.
- In our previous coin tossing example, we have

x	$p(x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$
Total	1

What does this notation mean?

Let A be some subset of the range of a discrete random variable. For example $A = \{\text{is greater than } 2\}$, $A = \{\text{between 0 and } 2\}$. We write

$$\{e \in S | X(e) \in A\} \tag{1}$$

for the set of outcomes e in S such that, $X(e)$, the value of the random variable is in the subset A. We typically write (**??**) as $(X \in A)$. For example,
$\{X \text{ is greater than } 2\} = (X > 2) = \{e \in S | X(e) > 2\}$
$\{X \text{ is between 0 and } 2\} = (0 < X < 2) = \{e \in S | 0 < X(e) < 2\}$.

Discrete Random Variables

Definition

A random variable $X : S \to \mathbb{R}$ is said to be discrete if its range,

$$\{X(e) | e \in S\},$$

is either finite or infinite countable, i.e., X can take on only a finite number- or a countable infinite number- of possible values x.

Probability Function

Definition

Let $X(S) \subset \mathbb{R}$ be a random variable. The *Probability function* of X, $\quad p : X(S) \to [0, 1]$ is defined as

$$p(x) = P(\{e \in S | X(e) = x\})$$

for any $x \in X(S)$. Thus, $p(x)$ is the probability that X equals x, and we write

$$p(x) = P(X = x).$$

The probability function, $p(x)$, assigns probability to each value x of X so that the following conditions are satisfied:

- $P(X = x) = p(x) \geq 0$ for all $x \in X(S)$
- $\sum_{x \in X(S)} P(X = x) = 1$, where the sum is over all possible values of x.

Cumulative Probability Distribution (CDF)

Definition

The *(cumulative) distribution function*, $F(b)$, for a random variable X evaluated at b is defined by

$$F(b) = P(X \leq b).$$

If X is discrete,

$$F(b) = \sum_{x=-\infty}^{b} p(x),$$

where $p(x)$ is the probability function.

Cumulative Probability Distribution (CDF)

Let A be some subset of the range of a discrete random variable. We have

- $\overline{(X \in A)} = (X \in \overline{A})$.

Cumulative Probability Distribution (CDF)

Let A be some subset of the range of a discrete random variable. We have

- $\overline{(X \in A)} = (X \in \overline{A})$.
- For sets A_1, A_2,...,

$$\cup_i (X \in A_i) = (X \in \cup_i A_i)$$

.

Cumulative Probability Distribution (CDF)

Let A be some subset of the range of a discrete random variable. We have

- $\overline{(X \in A)} = (X \in \overline{A})$.
- For sets A_1, A_2,...,

$$\cup_i (X \in A_i) = (X \in \cup_i A_i)$$

.

- $P(X > x) = 1 - P(X \leq x)$

Cumulative Probability Distribution (CDF)

Let A be some subset of the range of a discrete random variable. We have

- $\overline{(X \in A)} = (X \in \overline{A})$.
- For sets A_1, A_2,...,

$$\cup_i (X \in A_i) = (X \in \cup_i A_i)$$

.

- $P(X > x) = 1 - P(X \leq x)$
- If $a < b$, then $(X \leq b) \setminus (X \leq a) = (a < X \leq b)$

Cumulative Probability Distribution (CDF)

Let A be some subset of the range of a discrete random variable. We have

- $\overline{(X \in A)} = (X \in \overline{A})$.
- For sets A_1, A_2,...,

$$\cup_i (X \in A_i) = (X \in \cup_i A_i)$$

.

- $P(X > x) = 1 - P(X \leq x)$
- If $a < b$, then $(X \leq b) \setminus (X \leq a) = (a < X \leq b)$
-

$$\begin{aligned} P(a < X \leq b) &= P((X \leq b) \setminus (X \leq a)) \\ &= P(X \leq b) - P(X \leq a) = F(b) - F(a). \end{aligned}$$

Properties of the Distribution Function

A function, F, is a Cumulative Distribution Function iff

- $\lim_{x \to -\infty} F(x) = 0$

Properties of the Distribution Function

A function, F, is a Cumulative Distribution Function iff

- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to \infty} F(x) = 1$

Properties of the Distribution Function

A function, F, is a Cumulative Distribution Function iff

- $\lim_{x\to-\infty} F(x) = 0$
- $\lim_{x\to\infty} F(x) = 1$
- F is nondecreasing; that is if $a < b$, $F(a) \leq F(b)$.

Properties of the Distribution Function

A function, F, is a Cumulative Distribution Function iff

- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to \infty} F(x) = 1$
- F is nondecreasing; that is if $a < b$, $F(a) \leq F(b)$.
- F is right-hand continuous; that is

$$\lim_{x \to x_0^+} F(x) = F(x_0).$$

Example

Suppose a section of an electrical circuit has two relays, I and II, operating in parallel. The current will flow when a switch is thrown if either one or both of the relays close. The probability that a relay will close properly is 0.9. We assume that the relays operate independently. Let E_i denote the event that relay i closes properly when the switch is thrown. Then $P(E_i) = 0.9$. Let X be the number of relays that close properly. X can take the values $0, 1, 2$.

- The probability that 0 relays close is

$$
\begin{aligned}
p(0) &= P(X = 0) \\
&= P(\bar{E}_1 \cap \bar{E}_2) \\
&= P(\bar{E}_1)P(\bar{E}_2) \\
&= (0.1)(0.1) \\
&= 0.01.
\end{aligned}
$$

Example continue

- The probability that 1 relays close is

$$\begin{aligned} p(1) &= P(X = 1) \\ &= P(E_1 \bar{E}_2 \cup \bar{E}_1 E_2) \\ &= P(E_1 \bar{E}_2) + P(\bar{E}_1 E_2) \\ &= P(E_1)P(\bar{E}_2) + P(\bar{E}_1)P(E_2) \\ &= (0.9)(0.1) + (0.1)(0.9) = 0.18 \end{aligned}$$

- The probability that 2 relays close is

$$\begin{aligned} p(2) &= P(X = 2) \\ &= P(E_1 \cap E_2) \\ &= P(E_1)P(E_2) \\ &= (0.9)(0.9) = 0.81. \end{aligned}$$

Example continue

The current will flow if X is equal to at least 1; that is $X \geq 1$. This event has probability,

$$\begin{aligned} P(X \geq 1) &= P((X = 1) \cup (X = 2)) \\ &= P(X = 1) + P(X = 2) \\ &= 0.18 + 0.81 \\ &= 0.99. \end{aligned}$$

Example continue

We have

- $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = 0.01 + 0.18 = 0.19$

Example continue

We have

- $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = 0.01 + 0.18 = 0.19$
- Notice that $P(X < 2) = P(X \leq 1.5) = P(X \leq 1)$

Example continue

We have

- $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = 0.01 + 0.18 = 0.19$
- Notice that $P(X < 2) = P(X \leq 1.5) = P(X \leq 1)$
- $F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.01 + 0.18 + 0.81 = 1$

Example continue

We have

- $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = 0.01 + 0.18 = 0.19$
- Notice that $P(X < 2) = P(X \leq 1.5) = P(X \leq 1)$
- $F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.01 + 0.18 + 0.81 = 1$
- Notice that $F(k) = P(X \leq k) = P(X \leq 2) = F(2)$ for $k \geq 2$.

Example continue

We have

- $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = 0.01 + 0.18 = 0.19$
- Notice that $P(X < 2) = P(X \leq 1.5) = P(X \leq 1)$
- $F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.01 + 0.18 + 0.81 = 1$
- Notice that $F(k) = P(X \leq k) = P(X \leq 2) = F(2)$ for $k \geq 2$.
- Distribution Function:

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.01, & 0 \leq x < 1 \\ 0.19, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

Example continue

The height of the step is the probability associated with the value of x. Notice that we have

- $p(1) = P(0 < X \leq 1) = F(1) - F(0) = 0.19 - 0.01 = 0.18$
- $p(2) = P(1 < X \leq 2) = F(2) - F(1) = 1 - 0.19 = 0.81$
- Notice that $\lim_{x \to 1^+} F(x) = 0.19 = F(1)$. So F is right-hand continuous.
- $\lim_{x \to 1^-} F(x) = 0.01 \neq 0.19 = F(1)$. So F is not left-hand continuous.

Example

The distribution function of X, which is the number of times per week a student at a large university exercises, is given as follows,

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.30, & 0 \leq x < 1 \\ 0.50, & 1 \leq x < 2 \\ 0.80, & 2 \leq x < 3 \\ 0.90, & 3 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

Verify that F is a distribution function and find the probability function associated with F.

Solution

Solution

x	$p(x)$
0	$0.30 - 0 = 0.30$
1	$0.50 - 0.30 = 0.20$
2	$0.80 - 0.50 = 0.30$
3	$0.90 - 0.80 = 0.10$
4	$1 - 0.90 = 0.10$

Example

- Suppose that you are playing a game against the computer and the probability of winning a play is p. Let X be the number of unsuccessful trials before the first win. Assume that the events of play are independent.

Example

- Suppose that you are playing a game against the computer and the probability of winning a play is p. Let X be the number of unsuccessful trials before the first win. Assume that the events of play are independent.
- The probability of x unsuccessful plays before the first win is

$$p(x) = P(X = x) = (1-p)^x p \quad \text{for} \quad x = 0, 1, 2, ...$$

Example

- Suppose that you are playing a game against the computer and the probability of winning a play is p. Let X be the number of unsuccessful trials before the first win. Assume that the events of play are independent.
- The probability of x unsuccessful plays before the first win is

$$p(x) = P(X = x) = (1-p)^x p \quad \text{for} \quad x = 0, 1, 2, ...$$

- The cumulative distribution function is:

$$\begin{aligned} F(x) &= p(0) + p(1) + ... + p(x) \\ &= p + (1-p)p + (1-p)^2 p + ... + (1-p)^x p \\ &= p\left(\frac{1-(1-p)^{x+1}}{p}\right) = 1 - (1-p)^{x+1}. \end{aligned}$$

Example

Suppose that the probability of winning is $p = 0.10$ in the previous example.

(A) What is the probability that one has to play at most 3 unsuccessful games before the first win?

(B) What is the probability that one has to play at least 4 unsuccessful games before the first win?

(C) How many games do you need to play in order for the probability of winning at least once is 0.95 or greater?

Example

Suppose that the probability of winning is $p = 0.10$ in the previous example.
(A) What is the probability that one has to play at most 3 unsuccessful games before the first win?
(B) What is the probability that one has to play at least 4 unsuccessful games before the first win?
(C) How many games do you need to play in order for the probability of winning at least once is 0.95 or greater?

Solution

(A) $F(3) = P(X \leq 3) = 1 - (0.90)^{3+1} = 0.3439$.
(B) $P(X > 3) = 1 - F(3) = 0.90^4 = 0.6561$.

Solutions continue

- In part (B), we can see that we have a 65.61% probability of losing at least 4 times before the first win. In other words, if you play four games in a row, the probability of losing all 4 times is 0.6561 and the probability of winning at least once in the 4 plays is $F(3) = 0.3439$.

Solutions continue

- In part (B), we can see that we have a 65.61% probability of losing at least 4 times before the first win. In other words, if you play four games in a row, the probability of losing all 4 times is 0.6561 and the probability of winning at least once in the 4 plays is $F(3) = 0.3439$.
- Thus, in general, $F(x)$ is the probability that you win at least once in $x + 1$ successive plays.

Solutions continue

Solution

(C) $F(x-1) = 1-(0.90)^x$ *is the probability that you win at least once in* x *successive plays. Now let*

$$1-(0.9)^x = 0.95.$$

Then

$$\ln(0.90)^x = \ln(0.05)$$

and hence

$$x = \frac{\ln(0.05)}{\ln(0.90)} \approx 28.43.$$

Thus, you need to play 29 *times for the probability to be at least* 95% *that you win at least once.*

R-code

- In the previous example, the terms for the probability mass function, $p(x) = p(X = x) = (1 - p)^x p$, formed a geometric sequence. X is therefore called a *geometric random variable*.
- In R, we can calculate the probability mass function and the cumulative distributon function for the geometric random variable with $p = 0.1$ as follows:

  $> x < -c(0:5)$
  $> f < -dgeom(x, 1/10)$ # d indicates density
  $> F < -pgeom(x, 1/10)$ # p indicates probability
  $> data.frame(x, f, F)$
  $> y < -rgeom(100, 1/10)$ # simulate 100 geometric random variables
  $> hist(y)$

Example

Five lightbulb, 2 of which are defect are being mixed. Two of the lightbulb are randomly selected. Let X denote the number of defective lightbulb picked.
(A) Determine the probability mass function for X.
(B) Determine the Cumulative Distribution function for X.

Solution

- *The probability of selecting 0 defect lightbulbs is*
  $p(0) = \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}$

Example

Five lightbulb, 2 of which are defect are being mixed. Two of the lightbulb are randomly selected. Let X denote the number of defective lightbulb picked.
(A) Determine the probability mass function for X.
(B) Determine the Cumulative Distribution function for X.

Solution

- *The probability of selecting 0 defect lightbulbs is* $p(0) = \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}$
- *The probability of selecting 1 defect lightbulb is* $p(1) = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{3}{5}$

Example

Five lightbulb, 2 of which are defect are being mixed. Two of the lightbulb are randomly selected. Let X denote the number of defective lightbulb picked.
(A) Determine the probability mass function for X.
(B) Determine the Cumulative Distribution function for X.

Solution

- *The probability of selecting 0 defect lightbulbs is* $p(0) = \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}$
- *The probability of selecting 1 defect lightbulb is* $p(1) = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{3}{5}$
- *The probability of selecting 2 defect lightbulb is* $p(2) = \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{1}{10}$.

Example

Five lightbulb, 2 of which are defect are being mixed. Two of the lightbulb are randomly selected. Let X denote the number of defective lightbulb picked.
(A) Determine the probability mass function for X.
(B) Determine the Cumulative Distribution function for X.

Solution

- *The probability of selecting 0 defect lightbulbs is* $p(0) = \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}$
- *The probability of selecting 1 defect lightbulb is* $p(1) = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{3}{5}$
- *The probability of selecting 2 defect lightbulb is* $p(2) = \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{1}{10}$.

Solution continue

The probability mass function is given by,

x	$p(x)$
0	$\frac{3}{10}$
1	$\frac{3}{5}$
2	$\frac{1}{10}$

(B) The cumulative distribution function is

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.30, & 0 \leq x < 1 \\ 0.90, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$