Special Continuous distributions, chapter 3.3

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The Uniform Distribution

Consider experiments that consists of observing

- events in a certain time interval such as phone calls coming into a call center
- particles that have a certain diameter
- distances of a point from the beginning of a line.
- Suppose that we know that one such event has occurred in the interval (a, b).
- If X is the random variable for such experiments, we assume that X is equally likely to lie in any small subinterval within (a, b).
- X then has a uniform probability distribution.

Uniform distributions

Theorem

X has a uniform distribution on (a, b) if its p.d.f. is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b; \\ 0, & \textit{elsewhere} \end{cases}$$

We have

$$E(X) = \frac{a+b}{2} \quad \textit{and} \quad V(X) = \frac{(b-a)^2}{12}.$$

We say that X is $U(a, b)$.

The Uniform Distribution

Theorem

The distribution function is:

$$F(x) = \begin{cases} 0 & x < a; \\ \int_a^x \frac{1}{b-a} dt = \frac{x-a}{b-a}, & a \le x \le b; \\ 1, & x > b \end{cases}$$

If $(c, c+d) \subset (a, b)$, then

$$P(c \le x \le c+d) = P(x \le c+d) - P(x \le c)$$
$$= F(c+d) - F(c) = \frac{c+d-a}{b-a} - \frac{c-a}{b-a} = \frac{d}{b-a}.$$

Thus, the probability depends only on the length d of the subinterval.

Uniform probability density

Example

A customer is entering a store any time in a 5 minute interval. Let X be the time in minutes to the customer arrives. X is uniformly distributed. What is the distribution function, the expectation and the variance of X?

Solution

Solution

Uniform distribution:

$$f(x) = \begin{cases} \frac{1}{5} & 0 \leq x \leq 5; \\ 0, & otherwise \end{cases}$$

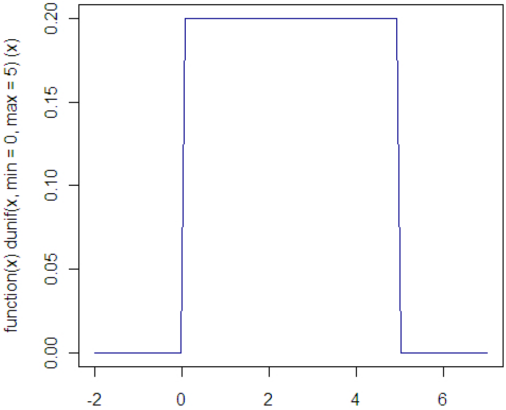
$$F(x) = \begin{cases} 0 & x < 0; \\ \int_0^x \frac{1}{5} dt = \frac{x}{5}, & 0 \leq x \leq 5; \\ 1, & x > 5 \end{cases}$$

$a = 0, \ b = 5.$

$$E(X) = \frac{a+b}{2} = \frac{5}{2} \ \text{and} \ V(X) = \frac{(b-a)^2}{12} = \frac{25}{12}.$$

Uniform probability density

In R: plot(function(x) dunif(x,min=0,max=5),-2,7)



Uniform probability distribution

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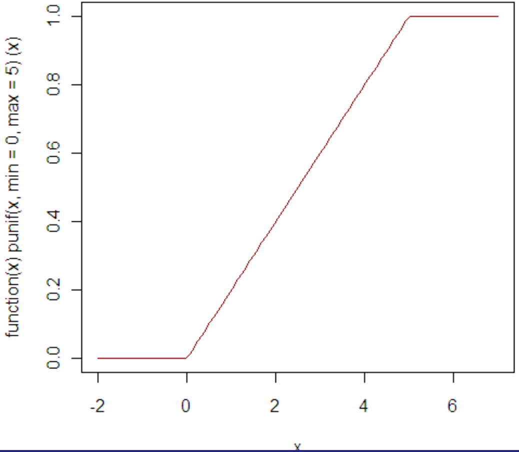

The Uniform distribution

Example

Suppose that X has a uniform distribution on the interval $(0, a)$ for $a > 1$. Find $P(X > X^2)$.

The Uniform distribution

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Solution

$X > X^2$ *is equivalent to* $0 < X < 1$ *so*

$$P(X > X^2) = P(0 < X < 1) = \int_0^1 \frac{1}{a} dx = \frac{1}{a}.$$

The exponential distribution

- The Probability over intervals of constant length decreases as the intervals moves further to the right.
- The life lengths of batteries.
- The weekly rainfall total for a section of a state.
- Time between successive hurricanes.

The exponential distribution function

Theorem

The random variable, X, has an exponential distribution if its p.d.f is

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, & x \geq 0; \\ 0, & \textit{elsewhere} \end{cases}$$

for $\theta > 0$. We have

$$E(X) = \theta \;\; \textit{and} \;\; Var(X) = \theta^2.$$

The parameter $\theta > 0$ is a constant that determines the rate at which the curve decreases.

The exponential distribution function

The cumulative distribution function:

$$F(x) = \begin{cases} P(X \le x) = \int_0^x \frac{1}{\theta} e^{-\frac{t}{\theta}}\,dt = 1 - e^{-\frac{x}{\theta}}, & x \ge 0; \\ 0, & \text{elsewhere.} \end{cases}$$

The exponential distribution function

Example

The life time, X, of a certain types of light bulbs follows an exponential distribution with a mean of 100 hours. Plot the probability density function and the distribution function for X.

Solution

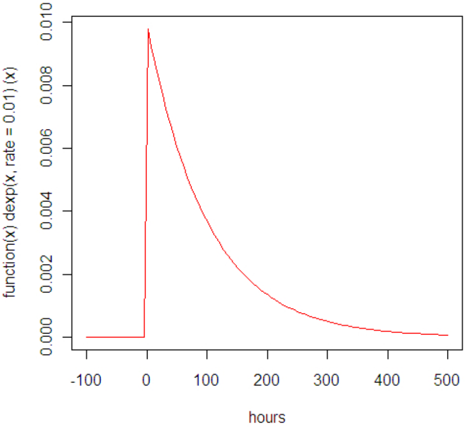
$\theta = 100.$

$$f(x) = \begin{cases} \frac{1}{100} e^{-\frac{x}{100}}, & x \geq 0; \\ 0, & elsewhere \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{100}}, & x \geq 0; \\ 0, & elsewhere. \end{cases}$$

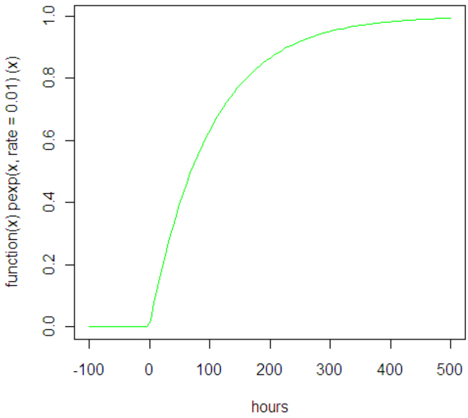
The exponential probability density function

plot(function(x) dexp(x,rate=0.01), -100, 500)



The exponential cumulative distribution function

plot(function(x) pexp(x,rate=0.01), -100, 500)



The Gamma function

Definition

The Gamma function is defined by

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-t} \, dy$$

for $t > 0$.

The Gamma function, Γ, has the following properties:

- $\Gamma(t+1) = t\Gamma(t)$
- $\Gamma(n) = (n-1)!$ for any integer $n \geq 1$.
- $\Gamma(n) = \infty$ for any integer $x \leq 0$.
- $\Gamma(t)$ is infinitely differentiable.
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Let X be an exponential random variable with mean θ. Let, $y = \frac{1}{\theta}x$ such that $dy = \frac{1}{\theta}dx$. Then

$$\begin{aligned} E(X^k) &= \frac{1}{\theta}\int_0^\infty x^k e^{-\frac{x}{\theta}}\, dx && (1)\\ &= \int_0^\infty (y\theta)^k e^{-y}\, dy && (2)\\ &= \theta^k \int_0^\infty y^k e^{-y}\, dy && (3)\\ &= \theta^k \Gamma(k+1) && (4)\\ &= \theta^k k!. && (5) \end{aligned}$$

The expectations and the variance of the exponential distribution

- Letting $k = 1$ in (1), we obtain,

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{0}^{\infty} x\frac{1}{\theta}e^{-\frac{x}{\theta}}\,dx = \theta.$$

The expectations and the variance of the exponential distribution

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- Letting $k = 2$ in (1), we obtain,

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{0}^{\infty} x^2\frac{1}{\theta}e^{-\frac{x}{\theta}}\,dx = 2\theta^2.$$

The expectations and the variance of the exponential distribution

- Letting $k = 1$ in (1), we obtain,

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{0}^{\infty} x\frac{1}{\theta}e^{-\frac{x}{\theta}}\,dx = \theta.$$

- Letting $k = 2$ in (1), we obtain,

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- Hence

$$V(X) = E(X^2) - \mu^2 = 2\theta^2 - \theta^2 = \theta^2.$$

The exponential distribution, example

Example

Suppose the number of phone calls passing through a particular cellular relay system, has a mean rate of 3 calls during a 2-min period. Let X denote the length of time in minutes between two successive phone calls and assume that X follows an exponential distribution.

(A) Find the mean time between two successive phone calls.

(B) Find the variance of the time between two successive phone calls.

(C) Find the median time between two successive phone calls.

(D) Find the probability that at least three minutes will pass before a call is passed through the relay system.

The exponential distribution, example

Solution

(A) The mean number of phone calls per minute is $\frac{3}{2}$. *The length of time between two successive phone calls has a rate of* $\frac{1}{\theta} = \frac{3}{2}$ *per minute. The mean time is* $E(X) = \theta = \frac{2}{3}$

(B) The variance is $Var(X) = \theta^2 = \frac{4}{9}$

(C) The p.d.f of X is $f(x) = \frac{3}{2}e^{-\frac{3}{2}x}$ *for* $x \geq 0$

and the c.d.f of X is $F(x) = 1 - e^{-\frac{3}{2}x}$.

To find the median time, we must solve $0.5 = e^{-\frac{3}{2}x}$. *Hence, the median time is,* $m = x = -\frac{2}{3}\ln(0.5)$.

(D) $P(X \geq 3) = \frac{3}{2}\int_3^\infty e^{-\frac{3}{2}x}\,dx = e^{-\frac{9}{2}} \approx 0.011$.

The memoryless property

Example

The life time, X, of a certain electrical component follows an exponential distribution with a mean of 100 hours. Find the conditional probability that the electrical component will last for more than 500 minutes given that it lasted for more than 200 minutes.

The memoryless property

Example

The life time, X, of a certain electrical component follows an exponential distribution with a mean of 100 hours. Find the conditional probability that the electrical component will last for more than 500 minutes given that it lasted for more than 200 minutes.

Solution

$$P(X > 500 | X > 200) = \frac{P[(X > 500) \cap (X > 200)]}{P(X > 200)} = \frac{P(X > 500)}{P(X > 200)}$$

$$= \frac{e^{-\frac{500}{100}}}{e^{-\frac{200}{100}}} = e^{-3} = P(X > 300).$$

The memoryless property

Thus, the probability that it will last longer than an additional 300 min given that it has lasted 200 min, is the same as the probability that it would last longer than 300 min when first used. Thus, for such a component, there is no advantage in replacing components that are working.

The exponential distribution, example

Example

Suppose the life length of a certain brand of automobile tires follows an exponential distribution with a mean of 10000 miles.
(A) Find the probability that one of these tires, bought today, will last over 10000 miles.
(B) Find the probability that at least one of the four tires bought today will last over 10000 miles.

The exponential distribution, example

Solution

$\theta = 10000$. *Let X be the number of miles a tire last.*

- *(A)*

$$P(X \geq 10000) = \frac{1}{10000} \int_{10000}^{\infty} e^{-\frac{x}{10000}} \, dx = e^{-1} = 0.368.$$

The exponential distribution, example

Solution

$\theta = 10000$. *Let X be the number of miles a tire last.*

- *(A)*

$$P(X \geq 10000) = \frac{1}{10000} \int_{10000}^{\infty} e^{-\frac{x}{10000}}\, dx = e^{-1} = 0.368.$$

- *(B) Let Y be the number of tires that last more than 10000 miles. Y follows a Binomial distribution with probability of success, $p = 0.368$. We have*

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - (1 - 0.368)^4 = 0.840.$$

The Chi-square distribution

Theorem

X has a chi-square distribution with r degrees of freedom if it has a p.d.f. given by

$$f(x) = \begin{cases} \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, & x \geq 0; \\ 0, & \textit{elsewhere} \end{cases}$$

where r is a positive integer and $\Gamma(t)$ is the gamma function given by $\Gamma(t) = \int_0^\infty y^{t-1}e^{-y}\,dy$ for $t > 0$.
We have

$$E(X) = r \qquad \textit{and} \qquad Var(X) = 2r.$$

We say that X is $\chi^2(r)$.

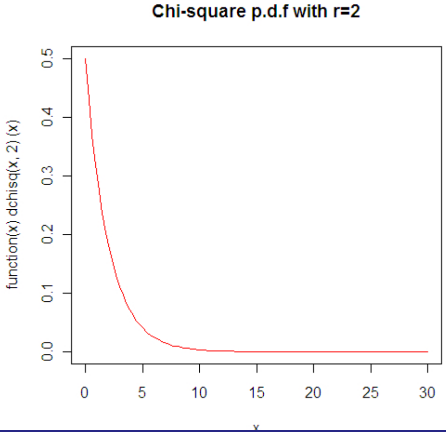
The Chi-square distribution

Notice that

- The mean is equal to the number of degrees of freedom.
- The variance is equal to twice the number of degrees of freedom.

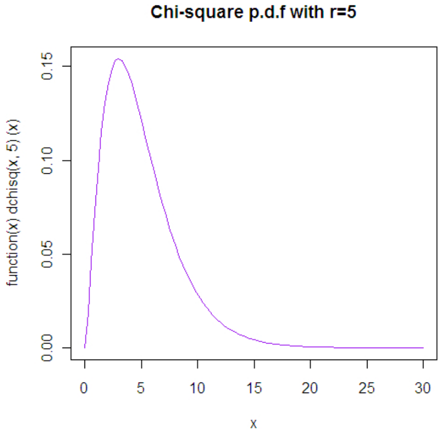
The Chi-square distribution

plot(function(x) dchisq(x,2),0, 30,main="Chi-square p.d.f with r=2",col="red")



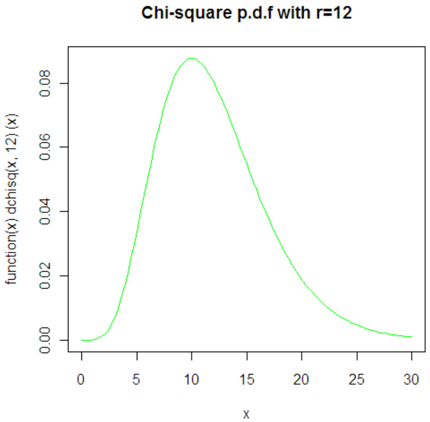
The Chi-square distribution

plot(function(x) dchisq(x,5),0, 30,main="Chi-square p.d.f with r=5",col="purple")



The Chi-square distribution

R-code: plot(function(x) dchisq(x,12),0, 30,main=”Chi-square p.d.f with r=12”,col=”green”)



The Chi-square distribution

- Table IV shows the values for $F(x) = \int_0^x \frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}} w^{\frac{r}{2}-1} e^{-\frac{w}{2}} \, dw$ for selected values of r and x.
- In the table, $\chi^2_\alpha(r)$ is the point on the x-axis with α probability to the right of it (area is α under the curve to the right of the point).

The Chi-square distribution,example

Example

Let X have a chi-square distribution with $r = 4$ degrees of freedom. Find

(A) $P(1.064 \leq X \leq 9.488)$

(B) $P(X > 11.14)$

Solution

(A)

$$
\begin{aligned}
P(1.064 \leq X \leq 9.488) &= F(9.488) - F(1.064) \\
&= 0.950 - 0.100 = 0.85
\end{aligned}
$$

(B) $P(X > 11.14) = 1 - F(11.14) = 1 - 0.975 = 0.25$

The Chi-square distribution, example

Example

If X is $\chi^2(8)$, determine the constants a and b such that
$P(a < X < b) = 0.95$
$P(X < a) = 0.025$

Solution

$a = \chi^2_{0.975}(8) = 2.18$
$b = \chi^2_{0.025}(8) = 17.54$