Definition of Probability, chapter 2.3

Grethe Hystad

September 4, 2013

Definition of Probability

Definition

A probability, P, is a function that assigns a real value to every event A so that the following axioms hold:

- $P(A) \geq 0$
- $P(S) = 1$
- If A_1, A_2,..., is a sequence of mutually exclusive events, (that is a sequence in which $A_i \cap A_j = \emptyset$ for any $i \neq j$), then $P\big(\cup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} P(A_i)$.

Probability rules

If follows that

- $P(\emptyset) = 0$
- $0 \leq P(A) \leq 1$
- If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
- $P(\bar{A}) = 1 - P(A)$
- If $A \subset B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Equally likely outcomes

- If S is a finite sample space and A is an event, then if each outcome is equally likely, the probability of A is $P(A) = \frac{\#(A)}{\#(S)}$.

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- **Example**

  Roll a symmetric dice. The probability of obtaining, i, where $i = 1, 2, 3, 4, 5, 6$ is
  $P(\{i\}) = \frac{1}{6}$.

Equally likely outcomes

- If $A \cap B = \emptyset$, then $\#(A \cup B) = \#(A) + \#(B)$ and hence $\frac{\#(A \cup B)}{\#(S)} = \frac{\#(A)}{\#(S)} + \frac{\#(B)}{\#(S)}$ or $P(A \cup B) = P(A) + P(B)$.

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  $P(A \cup B) = P(A) + P(B)$.

- **Example**

  Roll a dice.

  - (a) The probability of obtaining a "one" or a "three" is $P(\{1\} \cup \{3\}) = P(\{1\}) + P(\{3\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.
  - (b) The probability of obtaining an odd number is: $P(Odd) = P(\{1\} \cup \{3\} \cup \{5\}) =$
    $P(\{1\}) + P(\{3\}) + P(\{5\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

Example

Toss a coin 4 times. Let A be the event of "obtaining exactly 3 heads" Let B be the event of "obtaining exactly 4 heads".
(a) Find $P(A)$ and $P(B)$
(b) Find P(at least 3 heads)
(c) Find P(fewer than 3 heads)

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Solution

(a)

A={$HHHT$, $HHTH$, $HTHH$, $THHH$}

B={$HHHH$}

$\#(S) = 16$, $\#(A) = 4$ *and* $\#(B) = 1$

$P(A) = \frac{4}{16} = \frac{1}{4}$ *and* $P(B) = \frac{1}{16}$.

Solutions continue

Solution

- (b) *P(at least 3 heads)*=
  P(exactly 3 heads)+*P(exactly 4 heads)*= $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$.

Solutions continue

Solution

- *(b) P(at least 3 heads)=*
  P(exactly 3 heads)+P(exactly 4 heads)= $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$.
- *(c) P(fewer than 3 heads)=*
  1−*P(at least 3 heads)*=$1 - \frac{5}{16} = \frac{11}{16}$.

Example

If $P(A) = 0.3$, $P(B) = 0.4$, and $P(A \cap B) = 0.1$, find
(A)$P(A \cup B)$
(B)$P(A \cap \overline{B})$
(C)$P(\overline{A} \cup \overline{B})$

Example

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Solution

- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.1 = 0.6$
- $P(A \cap \overline{B}) = P(A) - P(A \cap B) = 0.3 - 0.1 = 0.2$

Example

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Solution

- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.1 = 0.6$
- $P(A \cap \overline{B}) = P(A) - P(A \cap B) = 0.3 - 0.1 = 0.2$
- $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - 0.1 = 0.9$

The following venn diagrams illustrates the previous problem and shows $(A \cup B)$, $(A \cap \overline{B})$, and $(\overline{A} \cup \overline{B})$.

venn3.jpg

Example

In a class of 60 students, 13 could not play an instrument, 17 are playing sport, and 10 could play an instrument and are playing sport. A student is randomly selected from this class. Find the probability that the selected student:

(a) Can play an instrument

(b) Cannot play an instrument and are playing sports.

(c) Can either play an instrument or are playing sports but not both.

Solution is given in class.