

Chapter 15

The Critical Region and the p-value

In this chapter, we will discuss the following topics:

- How to perform tests of hypotheses about the mean when the variance is known.
- How to compute the p-value with the R-function **pnorm**.
- We will look at the connection between the critical region, the test statistic, and the p-value.

Tests for a Population Mean and the P-value

Let X_1, X_2, \dots, X_n be a simple random sample of size n from a Normal distribution, $N(\mu, \sigma)$, where the population mean μ is unknown and the standard deviation σ is known. We wish to test the null hypothesis that μ has a particular value μ_0 , that is, $H_0 : \mu = \mu_0$. We calculate the one-sample test statistic with observed value

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$

and check whether this value is within or outside a critical region. The probability that z falls within the critical region when the null hypothesis is true is equal to a specified significance level which we denote by α . The significance level is typically set to 1% or 5%. If z falls within the critical region, then the null hypothesis is rejected at the chosen significance level.

The *p-value* associated with a test is the probability that we obtain a value of the test statistic that is at least as extreme as the observed value of our test statistic assuming the null hypothesis is true [1].

- If the p-value is less than the significance level, α , we reject H_0 and say that the result is *statistically significant* at the level α .
- The smaller the p-value, the stronger the evidence we have against the null hypothesis, H_0 .
- Instead of selecting the critical region in advance, we can report the p-value and consider all the values of α that would result in the rejection of the null hypothesis.

We have three possible alternative hypotheses:

- If the alternative hypothesis is that μ is greater than the specified value μ_0 , that is, $H_a : \mu > \mu_0$, then the p-value is the probability that we obtain a value of the test-statistic $Z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$ that is greater than or equal to the observed value z of the test-statistics assuming that the null hypothesis is true;

$$\text{p-value} = P(Z \geq z).$$

We reject H_0 when the p-value is smaller than α . If we choose the critical value z^* on the standard normal curve such that $P(Z \geq z^*) = \alpha$, then the critical region consists of the set of observed values z of the test statistic that is greater than or equal to the critical value z^* , that is, $z \geq z^*$. This is equivalent to $\bar{x} \geq \mu_0 + z^* \frac{\sigma}{\sqrt{n}}$. We illustrate the correspondence between the p-value and the critical region in the figure below. Notice that we do not need to standardize the random variable \bar{X} when we use R, since the p-value also can be calculated as

$$\text{p-value} = P(\bar{X} \geq \bar{x}).$$

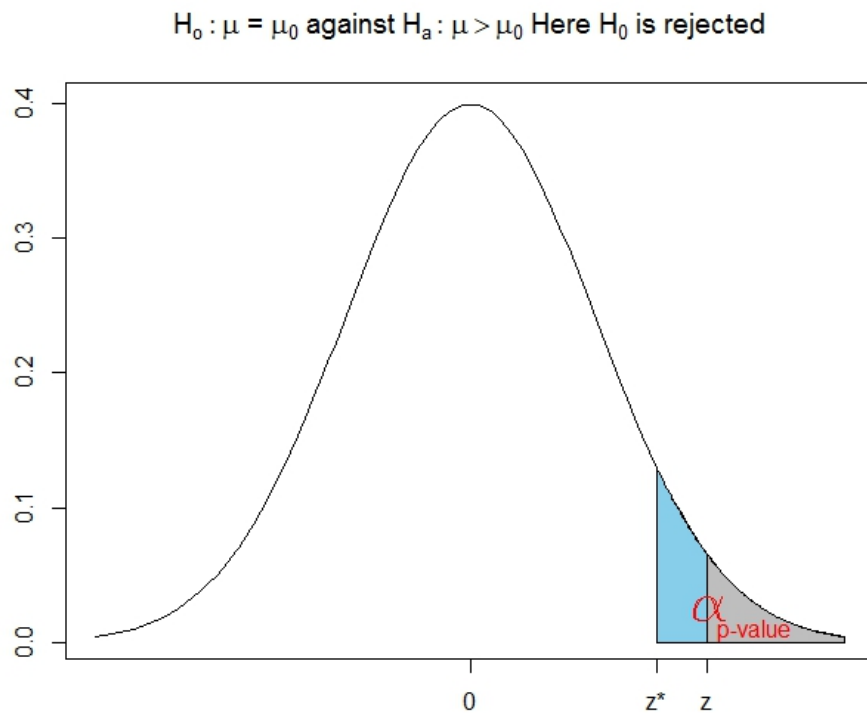


Figure 1: H_0 is rejected since $z \geq z^*$ and p-value $< \alpha$
p-value = area of dark shaded region
 α = area of both dark and blue shaded regions combined

Thus, we reject H_0 at the significance level α when the observed value z is such that $z \geq z^*$ or alternatively, when the p-value $< \alpha$.

- If the alternative hypothesis is that μ is smaller than the specified value μ_0 , that is, $H_a : \mu < \mu_0$, then the p-value is given by

$$\text{p-value} = P(Z \leq z) \text{ or equivalently } \text{p-value} = P(\bar{X} \leq \bar{x}).$$

Here the critical region consists of the set of all z for which $z \leq -z^*$ or $\bar{x} \leq \mu_0 - z^* \frac{\sigma}{\sqrt{n}}$, where z^* is chosen such that $P(Z \leq -z^*) = \alpha$ as illustrated in the figure below. Thus, we reject H_0 at the significance level α when the observed value z is such that $z \leq -z^*$ or alternatively, when the p-value $< \alpha$.

$H_0 : \mu = \mu_0$ against $H_a : \mu < \mu_0$ Here H_0 is rejected

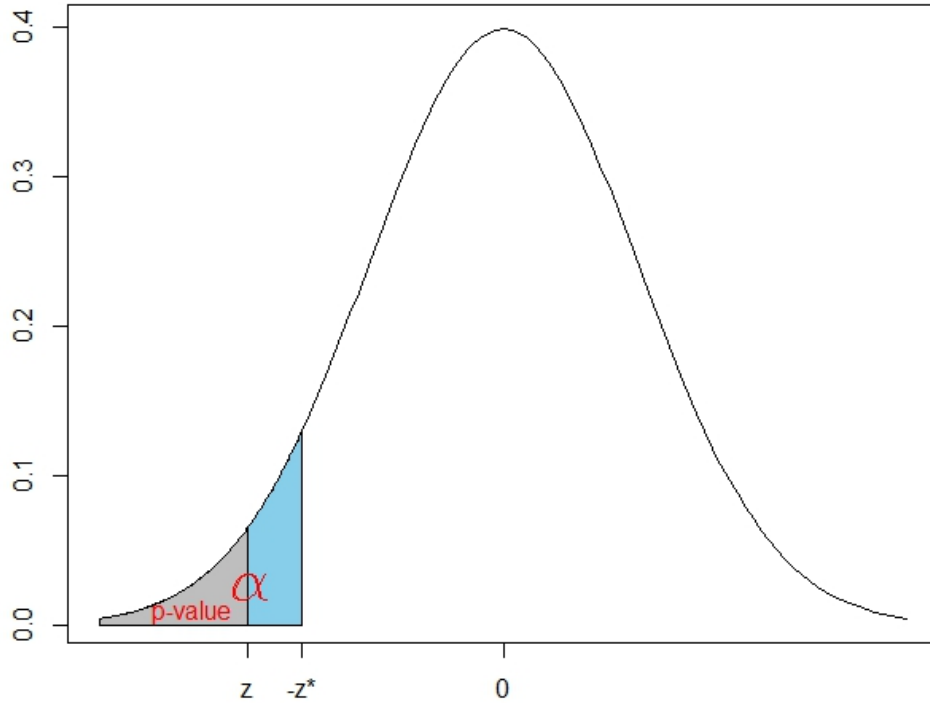


Figure 2: H_0 is rejected since $z \leq -z^*$ and p-value $< \alpha$
p-value=area of dark shaded region
 α =area of both dark and blue shaded regions combined

- If the alternative hypothesis is that μ is different from the specified value μ_0 , that is, $H_a : \mu \neq \mu_0$, then the p-value is given by

$$\text{p-value} = 2P(Z \geq |z|) \text{ or equivalently } \text{p-value} = 2P(\bar{X} - \mu_0 \geq |\bar{x} - \mu_0|).$$

The p-value is the area below $-z$ and above z as illustrated with the dark shaded regions in the figure below. Here the critical region consists of the set of all z for which $|z| \geq z^*$ or $|\bar{x} - \mu_0| \geq z^* \frac{\sigma}{\sqrt{n}}$. The critical value z^* is chosen such that $P(Z \geq z^*) = \frac{\alpha}{2}$ and $P(Z \leq -z^*) = \frac{\alpha}{2}$ so hence $P(|Z| \geq z^*) = \alpha$. Thus, we reject H_0 at the significance level α when the observed value z is such that $|z| \geq z^*$ or alternatively, when the p-value $< \alpha$.

$H_0: \mu = \mu_0$ against $H_a: \mu \neq \mu_0$ Here H_0 is rejected

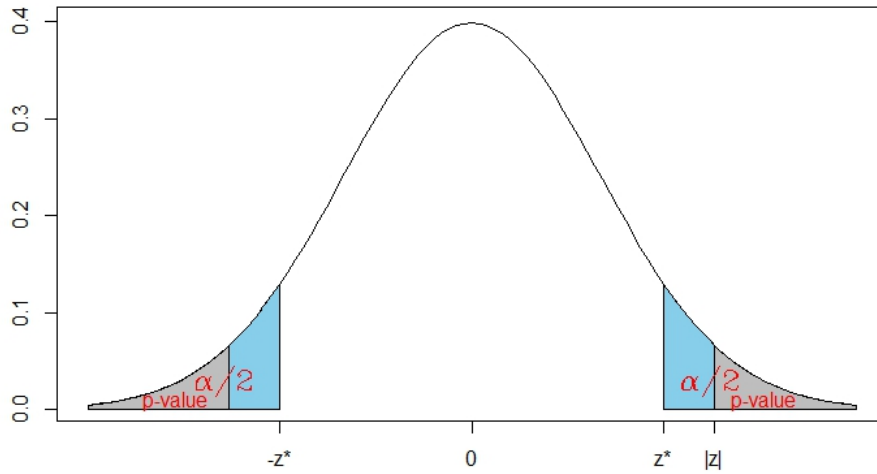
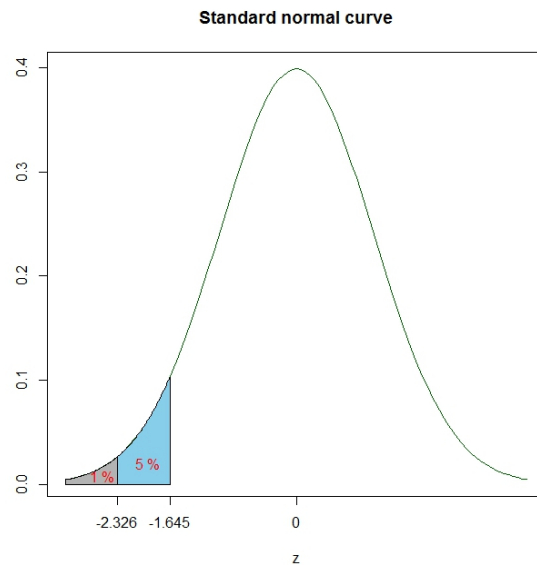


Figure 3: H_0 is rejected since $|z| \geq z^*$ and p-value $< \alpha$
 p-value=area of dark shaded regions
 α =area of both dark and blue shaded regions combined

The graph below illustrates the difference between significance at 1% level and at 5% level: The significance at 1% level includes the dark shaded region ($z < -2.326$). The significance



at the 5% level includes both the blue and dark shaded region ($z < -1.645$). We can see from the graph that significance at the 1% level implies significance at the 5% level but the converse is not true.

Problem. The American College Testing (ACT) exam is a standardized test for college admission in the U.S. The mean mathematics ACT score is $\mu = 20$ and the standard deviation is $\sigma = 5$. A particular high school claims that their school have a higher mean than 20. The school distributes the ACT test randomly to 25 seniors. Assume that it is known that the scores at the school are Normally distributed with standard deviation of 5.

(a) Assume the sample mean of the 25 seniors is $\bar{x} = 21$. Determine the p-value and whether the result is statistically significant at the $\alpha = 0.05$ level and at the $\alpha = 0.01$ level.

(b) Assume the sample mean of the 25 seniors is $\bar{x} = 22$. Determine the p-value and whether the result is statistically significant at the $\alpha = 0.05$ level and at the $\alpha = 0.01$ level.

Solution to part (a). We wish to test the null hypothesis that the population mean μ is 20 against the alternative hypothesis that μ is greater than 20:

$$H_0 : \mu = 20 \text{ against } H_a : \mu > 20.$$

If the null hypothesis is true, \bar{X} follows a normal distribution with mean 20 and standard deviation $\frac{5}{\sqrt{25}} = 1$. The p-value= $P(\bar{X} \geq 21)$. We obtain:

```
> pnorm(21,20,1, lower.tail=FALSE)
[1] 0.1586553
```

The p-value= 0.159. This is not significant at either $\alpha = 0.05$ or $\alpha = 0.01$. Alternatively, we could have standardized \bar{X} such that

$$P(\bar{X} \geq 21) = P\left(Z \geq \frac{21 - 20}{1}\right) = P(Z \geq 1).$$

In R:

```
> pnorm(1,lower.tail=FALSE)
[1] 0.1586553
```

Solution to part (b). The p-value= $P(\bar{X} \geq 22)$. We obtain:

```
> pnorm(22,20,1, lower.tail=FALSE)
[1] 0.02275013
```

The p-value= 0.023. This is significant at $\alpha = 0.05$ but not at $\alpha = 0.01$ since

$$0.01 < \text{p-value} < 0.05.$$

Alternatively, we could have standardized \bar{X} such that

$$P(\bar{X} \geq 22) = P\left(Z \geq \frac{22 - 20}{1}\right) = P(Z \geq 2).$$

In R:

```
> pnorm(2,lower.tail=FALSE)
[1] 0.02275013
```

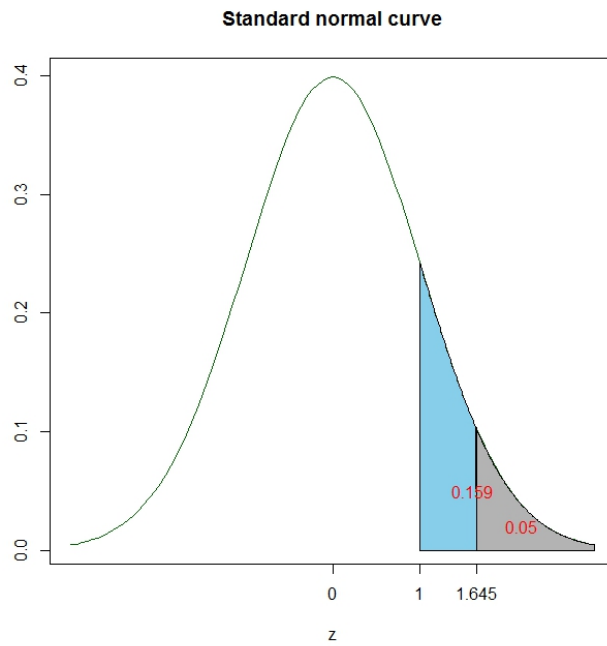


Figure 4: $p\text{-value} = 0.159 > 0.05$ so H_0 is not rejected

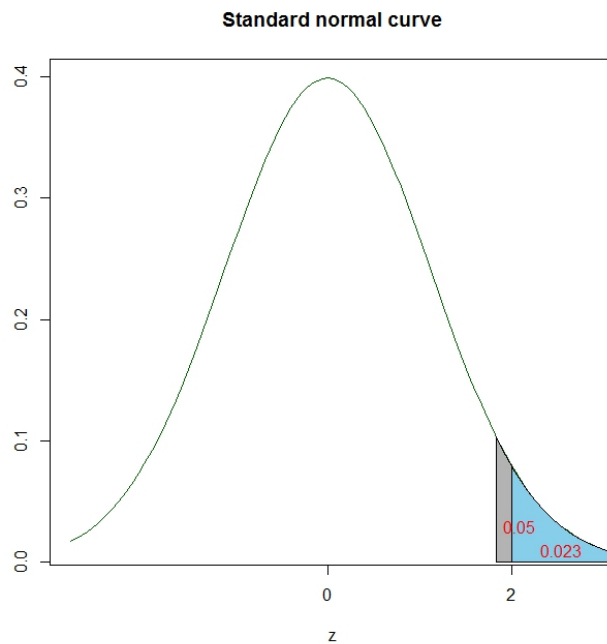


Figure 5: $p\text{-value} = 0.023 < 0.05$ so H_0 is rejected at the 0.05 significance level

The school have some evidence against the null hypothesis when $\bar{x} = 22$.

Explanation. The code can be explained as follows:

- Recall that the command `pnorm(x, μ, σ, lower.tail=FALSE)` computes $P(X \geq x)$.

Problem. A particular type of capacitor has resistance that follows a Normal distribution with a standard deviation of 200 megoohms. The following data shows 8 measurements of the resistance of the capacitors from this population. Assume the data is a simple random sample:

536.19 717.47 883.35 640.24 1029.02 505.32 756.86 812.76

The capacitor is said to have a mean resistance of 800 megoohms. Do the sample give good evidence that the true mean resistance is different from 800?

Solution. We wish to test the hypothesis:

$$H_0 : \mu = 800 \text{ against } H_a : \mu \neq 800.$$

If the null hypothesis is true, \bar{X} follows a normal distribution with mean 800 and standard deviation $\frac{200}{\sqrt{8}}$. We use R to calculate the test statistics $z = \left| \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \right|$ and the p-value which is given by $2P(Z \geq |z|)$.

```
> x=c(536.19, 717.47, 883.35, 640.24, 1029.02, 505.32, 756.86, 812.76)
> xbar=mean(x)
> xbar
[1] 735.1513
> sdx=200/sqrt(8)
> z=abs((xbar-800)/sdx)
> z
[1] 0.9170998
> y=pnorm(z,lower.tail=FALSE)
> y
[1] 0.1795452
> p_value=2*y
> p_value
[1] 0.3590903
```

The p-value= 0.359 > 0.05 so we do not reject the null hypothesis. Thus, we do not have evidence for that the true mean resistance is different from 800 megoohms.

Alternatively, we could have computed the p-value = $2P(\bar{X} - \mu_0 \geq |\bar{x} - \mu_0|)$ directly in R. Here we use the fact that $(\bar{X} - \mu_0)$ is Normal with mean 0 and standard deviation $\frac{200}{\sqrt{8}}$:

```
> 2*pnorm(abs(xbar-800),0,sdx,lower.tail=FALSE)
[1] 0.3590903
```

Explanation. The code can be explained as follows:

- The command `abs()` returns the absolute value of the expression to be calculated.

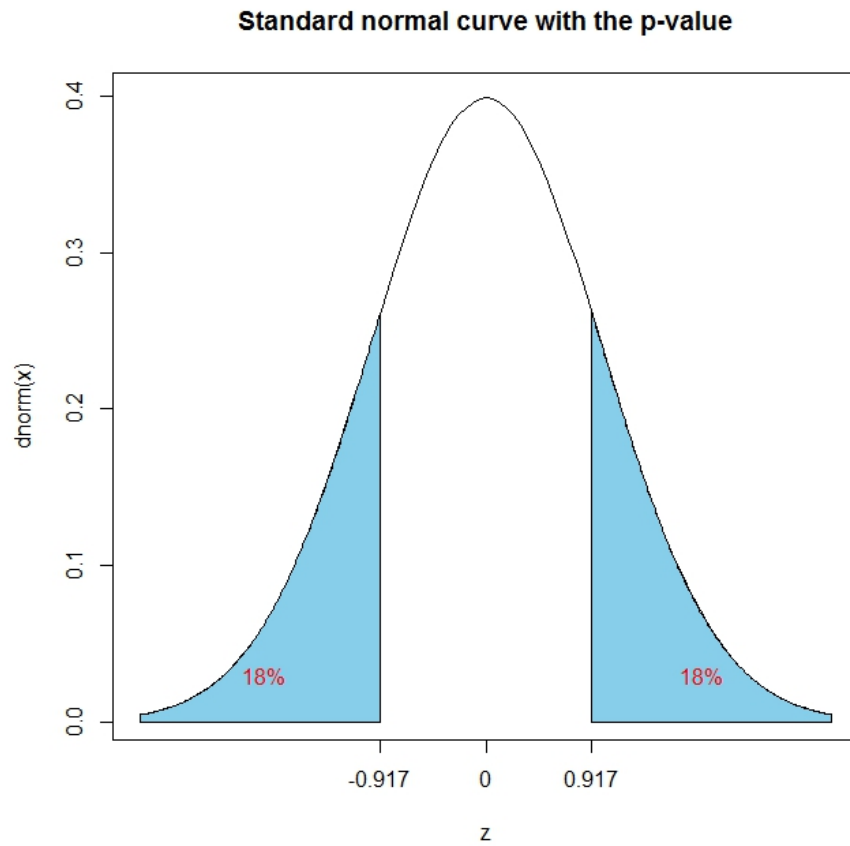


Figure 6: The p-value is 0.36.

References

- [1] E. A. Tanis, R. V. Hogg. *A Brief Course in Mathematical Statistics*. Pearson Prentice Hall, Upper Saddle River, NJ, 2008.

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