Bayes's Theorem, chapter 1.5

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Theorem of Total Probability and Bayes's Theorem

Definition

Events B_1, B_2,..., B_k are said to partition a sample space S if the following two conditions are satisfied:

- $B_i \cap B_j = \emptyset$ for any pair i and j with $i \neq j$.
- $B_1 \cup B_2 \cup \cdots \cup B_k = S$.

For example if the events B_1 and B_2 partition the sample space S, then for any event A in the sample space S,

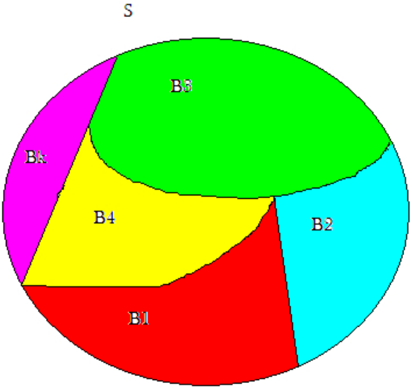
$$A = (A \cap B_1) \cup (A \cap B_2),$$

where $A \cap B_1$ and $A \cap B_2$ are mutually exclusive events. Thus

$$P(A) = P(A \cap B_1) + P(A \cap B_2) = P(B_1)P(A \mid B_1) + P(B_2)P(A \mid B_2).$$

Partition of the sample space

The following figure illustrates a partition B_1, B_2, B_3, B_4, and B_k of the sample space S.



Bayes's Theorem

Theorem

Theorem of Total Probability: If $B_1, B_2, ..., B_k$ is a collection of mutually exclusive and exhaustive events, then for any event A,

$$P(A) = \sum_{i=1}^{k} P(B_i)P(A \mid B_i).$$

Theorem

Bayes's Theorem: If the events $B_1, B_2, ..., B_k$ form a partition of the sample space S, and A is any event in S, then

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{\sum_{j=1}^{k} P(A \mid B_j)P(B_j)}.$$

Example from R. Schraffer's book

Example

A diagnostic test for a certain disease is said to be 90% accurate; that is, if a person has the disease, the test will detect it with probability 0.9. Moreover, if a person does not have the disease, the test will report that he or she doesn't have it with probability 0.9. Only 1% of the population has the disease in question. If the diagnostic test reports that a person chosen at random from the population has the disease, what is the conditional probability that the person does in fact, have the disease?

Solution

Solution

Let S, N, TS, and TN be the events,
S: has the disease
TS: positive test result
Then by Bayes's Theorem,

$$P(S \mid TS) = \frac{P(S \cap TS)}{P(TS)} = \frac{P(TS \mid S)P(S)}{P(TS \mid S)P(S) + P(TS \mid S')P(S')}$$
$$= \frac{0.90 * 0.01}{0.90 * 0.01 + 0.10 * 0.99} = 0.0833.$$

Problem

Lightbulbs are manufactored by three factories, I, II, and III.

- 10% *of the bulbs were made in factory I*
- 40% *of the bulbs were made in factory II*
- 50% *of the bulbs were made in factory III.*

Suppose that

- 2% *of the lightbulbs produced by I are defective*
- 5% *of the lightbulbs produced by II are defective*
- 3% *of the lightbulbs produced by III are defective.*

Assume that an lightbulb selected at random from the stockpile is observed to be defective.
What is the probability that the lightbulb came from factory I?

Solution given in class.

Problem

Suppose we have two baskets, basket I and II, which contains balls.
Basket I contains 7 *green balls and* 2 *red balls.*
Basket II contains 3 *green balls and* 5 *red balls.*
Select a basket at random and select two ball from its basket without replacing the first ball.
(A) What is the probability that the second ball selected is red?
(B) What is the probability that the second ball selected is green?

Solution given in class.

Partition of the sample space

The following figure illustrates a tree diagram for the previous problem.