Independent events, chapter 1.4

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Independence

Example

- Flip a coin twice and let A, B, and C be the events
  - A: both tosses are head.
  - B: heads on the first flip.
  - C: heads on the second flip.

$A = \{HH\}$ $\qquad B = \{HH, HT\}$ $\qquad C = \{TH, HH\}$

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- Notice $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2} = P(A)$. The occurrence of the event B has not affected the probability for the event A. We then say that the events A and B are independent.

Independence

If the knowledge that an event B has occurred, does not result in a change of the probability of A, that is $P(A \mid B) = P(A)$, the events A and B are said to be *independent*.

Definition

Two events A and B are said to be independent iff

$$P(A \cap B) = P(A)P(B).$$

If $P(B) > 0$, this is equivalent

$$P(A \mid B) = P(A) \quad \text{or}$$

if $P(A) > 0$

$$P(B \mid A) = P(B).$$

Multiplicative Rule

Theorem

Multiplicative Rule. If A and B are any two events, then

$$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B).$$

If A and B are independent, then

$$P(A \cap B) = P(A)P(B).$$

Example

Two dice are rolled and the number on the upper face is observed. Is the event that the sum of the numbers on the upper face is 7 independent of the event that the number observed on the second die is a 5?

Solution

Solution

Let A and B be the events,

- *A: sum of the numbers on the upper face is 7,*
- *B: number on the upper face on second die is 5.*
- $A = \{\{(1,6)\},\{(6,1)\},\{(2,5)\},\{(5,2)\},\{3,4\},\{4,3\}\}$,
- $B = \{\{(1,5)\},\{(2,5)\},\{(3,5)\},\{(4,5)\},\{(5,5)\},\{(6,5)\}\}$,
- $A \cap B = \{\{(2,5)\}\}$.
- *Total outcome is* $6 \times 6 = 36$.
- $P(A) = P(B) = \frac{6}{36} = \frac{1}{6}$
- $P(A \cap B) = \frac{1}{36}$
- $P(A)P(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = P(A \cap B)$ *so A and B are independent events.*

Example

Example

A card is drawn from a standard deck of 52 cards. Without replacing the first card, a second card is drawn. Is the event that the second card is a diamond independent of the event that the first card is a diamond?

Solution

Solution

- *Let A_i be the event that the i^{th} card is a diamond.*

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- *Let* A_i *be the event that the* i^{th} *card is a diamond.*
- *Since there are a total of* 13 *diamonds and* 12 *left if the first card drawn is a diamond, we have* $P(A_2 \mid A_1) = \frac{12}{51} = \frac{4}{17}$.

Solution

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- *Let A_i be the event that the i^{th} card is a diamond.*
- *Since there are a total of 13 diamonds and 12 left if the first card drawn is a diamond, we have* $P(A_2 \mid A_1) = \frac{12}{51} = \frac{4}{17}$.
- *Now*

$$\begin{aligned} P(A_2) &= P(A_1)P(A_2 \mid A_1) + P(\overline{A}_1)P(A_2 \mid \overline{A}_1) \\ &= \frac{13}{52}\frac{12}{51} + \frac{39}{52}\frac{13}{51} \\ &= \frac{1}{4}. \end{aligned}$$

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- *Let A_i be the event that the i^{th} card is a diamond.*
- *Since there are a total of 13 diamonds and 12 left if the first card drawn is a diamond, we have $P(A_2 \mid A_1) = \frac{12}{51} = \frac{4}{17}$.*
- *Now*

$$\begin{aligned} P(A_2) &= P(A_1)P(A_2 \mid A_1) + P(\overline{A}_1)P(A_2 \mid \overline{A}_1) \\ &= \frac{13}{52}\frac{12}{51} + \frac{39}{52}\frac{13}{51} \\ &= \frac{1}{4}. \end{aligned}$$

- *Hence $P(A_2 \mid A_1) \neq P(A_2)$, so A_2 and A_1 are not independent events.*

Example

The following table shows the percentage by age and gender categories of the 2004 U.S. population aged 16 and older:

Age (in years)	Men	Women	Total
16 to 24	18	17	35
25 to 44	15	14	29
45 to 64	12	12	24
65 and Older	5	7	12
Total	50	50	100

Table: Source: Introduction to Probability and its Applications by R. Scheaffer and U.S. Census Bureau.

Is age independent of gender?

Solution

Table: The conditional distribution of age groups for men and women.

Age (in years)	*Men*	*Women*
16 to 24	*0.36*	*0.34*
25 to 44	*0.30*	*0.28*
45 to 64	*0.24*	*0.24*
65 and Older	*0.10*	*0.14*

$P(16 \ \ to \ \ 24 \mid men) = 0.36 \approx P(16 \ \ to \ \ 24) = 0.35$
$P(25 \ \ to \ \ 44 \mid men) = 0.30 \approx P(15 \ \ to \ \ 44) = 0.29$
$P(45 \ \ to \ \ 64 \mid men) = 0.24 = P(15 \ \ to \ \ 44) = 0.24$
$P(65 \ \ and \ \ Older \ \ \mid men) = 0.10 \approx P(65 \ \ and \ \ Older) = 0.12$

Solution continue

Solution

$P(16 \ \ to \ \ 24 \mid women) = 0.34 \approx P(16 \ \ to \ \ 24) = 0.35$
$P(25 \ \ to \ \ 44 \mid women) = 0.28 \approx P(15 \ \ to \ \ 44) = 0.29$
$P(45 \ \ to \ \ 64 \mid women) = 0.24 = P(15 \ \ to \ \ 44) = 0.24$
$P(65 \ \ and \ \ Older \ \ \mid women) = 0.14 \approx P(65 \ \ and \ \ Older) = 0.12$
Hence the events gender and age are independent.

Independence

Theorem

A and B are independent events iff A' and B' are independent events.

Independence, Example

Example

Suppose three inspectors look at a product and that their probabilities of detecting a defect are 0.98, 0.97, and 0.95 respectively. Assume that the inspectors detect defects independently from each other.
(A) What is the probability that at least one inspector detect a defect?
(B) What is the probability that exactly two inspectors find defects in the product?

Solution, part A

Solution

(A)

- *Let A_i denote the event that the i^{th} inspector does not find a defect, where $i = 1, 2, 3$.*

Solution, part A

Solution

(A)

- *Let* A_i *denote the event that the* i^{th} *inspector does not find a defect, where* $i = 1, 2, 3$.
- *Then* $P(A_1) = 0.02$, $P(A_2) = 0.03$, $P(A_3) = 0.05$.

Solution, part A

Solution

(A)

- *Let A_i denote the event that the i^{th} inspector does not find a defect, where $i = 1, 2, 3$.*
- *Then $P(A_1) = 0.02$, $P(A_2) = 0.03$, $P(A_3) = 0.05$.*
- *The probability that no inspectors find a defect is $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) = (0.02)(0.03)(0.05)$.*

Solution, part A

Solution

(A)

- *Let A_i denote the event that the i^{th} inspector does not find a defect, where $i = 1, 2, 3$.*
- *Then $P(A_1) = 0.02$, $P(A_2) = 0.03$, $P(A_3) = 0.05$.*
- *The probability that no inspectors find a defect is $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) = (0.02)(0.03)(0.05)$.*
- *The probability that at least one inspector find a defect is $P[(A_1 \cap A_2 \cap A_3)'] = 1 - P(A_1 \cap A_2 \cap A_3) = 1 - (0.02)(0.03)(0.05) = 0.99997$.*

Solution, part B

Solution

(B) The probability that exactly two inspectors find the defect is

$$
\begin{aligned}
&P(A_1A_2'A_3') + P(A_1'A_2A_3') + P(A_1'A_2'A_3)\\
&= (0.02)(0.97)(0.95) + (0.98)(0.03)(0.95) + (0.98)(0.97)(0.05)\\
&= 0.09389
\end{aligned}
$$