Conditional Probability, chapter 1.3

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- Suppose that we are told at least one of the tosses is a head. What is the probability that both the tosses is a head? The sample space is now reduced to $S_R = \{HH, HT, TH\}$. Since the outcomes are equally likely, the probability that both tosses is a head is now $\frac{1}{3}$.

Conditional Probability, example

- Let A and B be the events
  - A: both tosses are head.
  - B: at least one of the tosses is a head.

$$S = \{HH, HT, TH, TT\} \qquad N(S) = 4$$
$$A = \{HH\} \qquad N(A) = 1$$
$$B = \{HH, HT, TH\} \qquad N(B) = 3$$
$$A \cap B = \{HH\} \qquad N(A \cap B) = 1$$

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- Notice

$$P(A \mid B) = \frac{1}{3} = \frac{N(A \cap B)/N(S)}{N(B)/N(S)} = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4}.$$

Conditional Probability

Definition

If A and B are any two events, then the conditional probability of A given B, denoted $P(A \mid B)$, is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

provided that $P(B) > 0$.

Notice:
If it is given that B occurs, this will affect the probability of A, and all the elementary events which do not occur should be excluded, and thus only the ones in B should be left.

Axioms of probability

Conditional probability satisfy the three axioms of probability:

- $0 \leq P(A \mid B) \leq 1$
- $P(S \mid B) = 1$
- If $A_1, A_2, ...,$ are mutually exclusive events, then so are $A_1 \cap B, A_2 \cap B$,..., and

$$P\Big(\cup_{i=1}^{\infty} (A_i \mid B) \Big) = \sum_{i=1}^{\infty} P(A_i \mid B).$$

Conditional Probability, example

The following table displays the results of a study, by educational level, of those who have smoked a cigarette within the past year for persons aged 26 and older.

Education	Smoked	Have not smoked	Total
< High School Diploma	10,393	19,472	29,865
High School Graduate	17,798	39,247	57,045
Some College	13,463	30,969	44,432
College Graduate	8,320	43,357	51,677
Total	49,974	133,045	183,019

Table: Source: Introduction to Probability and its Applications by R. Scheaffer.

Example continues

The percentage of people who has smoked a cigarette within the past year is $\frac{49{,}974}{183{,}019} * 100 = 27.3$.
The percentage of College Graduates who has smoked a cigarette within the past year is $\frac{8{,}320}{51{,}677} * 100 = 16.1$.

Example

What is the probability that a randomly selected person who has smoked a cigarette in the past year has completed college?

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Solution

Let A and B denote the events, A: completed College
B: smoked a cigarette in the past year.
Then $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{0.161 * 0.282}{0.273} = 0.166$.

Example from Introduction to Probability and its Applications by R. Scheaffer.

Example

Suppose that the probability that a student passes a test the first time is 0.8. For those who fail the first time, the probability of passing the test the second time is 0.6.
(A) Find the probability that a randomly selected student passes the test.
(B) If the student passes the test, what is the probability that she or he did so on the first try?

Solution

Solution

(A) Let A_i and B be the events,
A_i: student passes the test the i^{th} time.
B: student passes the test.

$$\begin{aligned} P(B) = P(A_1) + P(A_2 \cap A_1') &= P(A_1) + P(A_2 \mid A_1')P(A_1') \\ &= 0.80 + 0.60 * 0.20 \\ &= 0.92. \end{aligned}$$

Solution

Solution

(B)

$$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1)}{P(B)} = \frac{0.80}{0.92} = 0.87$$

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Solution

Let A be the event of threes spades in the first seven cards drawn. Let B be the event of a spade on the eighth draw.

Then $P(A) = \frac{\binom{13}{3}\binom{39}{4}}{\binom{52}{7}} = 0.176$

$P(B \mid A) = \frac{10}{45} = 0.222$. *Hence*
$P(A \cap B) = P(A)P(B \mid A) = (0.176)(0.222) = 0.039.$

Extension of multiplication rule

In the case of three events:

$$P(A \cap B \cap C) = P((A \cap B) \cap C) = P(A \cap B)P(C \mid A \cap B),$$

where

$$P(A \cap B) = P(A)P(B|A).$$

Hence

$$P(A \cap B \cap C) = P(A)P(B \mid A)P(C|A \cap B).$$

This argument can be extended to k events.

Extension of multiplication rule, Example

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Solution

Denote the event: H_1*=heart on first draw,* D_2*=diamond on second draw,* H_3*=heart on third draw. Then*
$P(H_1 \cap D_2 \cap H_3) = P(H_1)P(D_2 \mid H_1)P(H_3 \mid H_1 \cap D_2) = \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{12}{50}$

Problem 1.3-7 (Homework)

Example

An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

Problem 1.3-7 (Homework)

Example

An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

Solution

Hint: Let O_i be the event of drawing an orange ball on the i^{th} draw for $i = 1, 2$. Then we must find

$$P(O_1 \cap O_2 | O_1 \cup O_2) = \frac{P((O_1 \cap O_2) \cap (O_1 \cup O_2))}{P(O_1 \cup O_2)} = \frac{P(O_1 \cap O_2)}{P(O_1) + P(O_2) - P(O_1 \cap O_2)}$$

where $P(O_2) = P(O_2|O_1)P(O_1) + P(O_2|O_1')P(O_1')$