

# Methods of enumeration, 1.2

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# Multiplication Principle

## Theorem

*Fundamental Principle of Counting:*

*If the first task of an experiment can result in  $n_1$  possible outcomes and for each such outcome, the second task can result in  $n_2$  possible outcomes, then there are  $n_1 n_2$  possible outcomes for the two tasks together.*

# Multiplication Principle

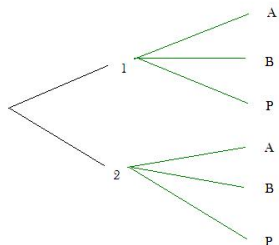
## Example

Suppose two patients, patient 1 and patient 2, are randomly selected to participate in an experiment. They receive either drug A, drug B, or a placebo P. The total possible outcomes is  $(2)(3) = 6$ . The outcomes of the experiment can be written as ordered pairs:  $(1, A)$ ,  $(1, B)$ ,  $(1, P)$ ,  $(2, A)$ ,  $(2, B)$ ,  $(2, P)$ .

# Multiplication Principle, example continues

## Example

We can also illustrate the outcomes with a tree-diagram:



# Example

## ■ Example

A box contains 4 balls, of which 1 is green and 3 are blue. Suppose two balls are selected at random. Let  $A$  denote the event that the two balls have different colors. Find  $P(A)$ .

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## ■ Solution

- *Number the balls, 1, 2, 3, 4, where ball 1 is green and balls 2, 3, 4 are blue.*
- *Total possible outcomes are  $4 \cdot 3 = 12$ .*
- *Because the balls are randomly selected, each of the 12 outcomes has a probability of  $\frac{1}{12}$ .*
- $P(A) = P(\{(1, 2)\} \cup \{(1, 3)\} \cup \{(1, 4)\} \cup \{(2, 1)\} \cup \{(3, 1)\} \cup \{(4, 1)\}) = \frac{6}{12} = \frac{1}{2}$ .

# Sampling with replacement, order matters, example 1

## ■ Example

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# Sampling with replacement, order matters, example 1

## ■ Example

Suppose you want to choose a four digit pin code for your debit card. Each digit can take the numbers 0, 1, ..., 9. How many possible choices do you have?

## ■ Solution

*Since each of the 10 digits could be chosen for each position, there are*

$$10 \times 10 \times 10 \times 10 = 10000$$

*possible choices.*

In general, if we have  $n$  objects and we select  $r$  of these with replacement, and order is important, then there are  $n^r$  possible outcomes.



## Sampling without replacement, order matters, example 2

### ■ Example

7 applicants have applied for 7 different jobs. How many ways can the jobs be filled.

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### Solution

- *Here the selection is without replacement because the same person does not fill more than one position.*
- *There are 7 choices for filling the first position, 6 for the second, ..., and 1 for the last position.*
- *By the Multiplication Principle, the positions can be filled in  $7 \times 6 \times 5 \cdots \times 1 = 7! = 5040$  ways.*

## Sampling without replacement, order matters, example 3

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## ■ Solution

- *Here the selection is without replacement because the same person does not fill more than one position.*
- *There are 7 choices for filling the first position, 6 for the second and 5 for the third.*
- *Can be filled in  $7 \times 6 \times 5 = 210$  ways.*

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## ■ Example

The number of permutations of the set  $\{1, 2, 3\}$  is  $3! = 6$ . They are 123 132 213 231 312 321.

# Sampling without replacement, order matters

## ■ Example

The number of 2-permutations of the set  $\{1, 2, 3\}$  is  $\frac{3!}{(3-2)!} = 6$ .  
They are 12 21 13 31 23 32.

# Sampling without replacement, order matters

## ■ Example

The number of 2-permutations of the set  $\{1, 2, 3\}$  is  $\frac{3!}{(3-2)!} = 6$ .  
They are 12 21 13 31 23 32.

## ■ Theorem

*Permutations: The number of ordered arrangements or permutations  ${}_n P_r$  of  $r$  objects selected from  $n$  distinct objects ( $r \leq n$ ) is given by*

$${}_n P_r = n(n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}.$$

In Example 3,  ${}_7 P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \times 6 \times 5 = 210$ .



# Combinations: Sampling without replacement, order does not matter

## Definition

A combination of a set of objects is a subset of those objects.

## Theorem

*Combinations: The number of distinct subsets or combinations of size  $r$  that can be selected from  $n$  distinct objects ( $r \leq n$ ) is given by*

$${}_n C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

*This is the same as the number of ways of partitioning  $n$  distinct objects into exactly two subsets containing  $r$  and  $(n - r)$  elements, respectively.*



# Combinations

In R, the command for  $\binom{n}{r}$  is: `choose(n,r)`

## Example

There are  $\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$  subsets of size 2 of the set  $\{1, 2, 3\}$ .  
They are  $\{1, 2\}$ ,  $\{1, 3\}$ ,  $\{2, 3\}$ .

- If order matters, use permutations.
- If order does not matters, use combinations.

# Combinations, Example

## ■ Example

In a Norway lottery, a player selects 6 numbers from 1 to 48. During the draw, 6 numbers are drawn at random and without replacement. To win the first prize, all 6 numbers must match those drawn in any order. How many winning numbers are possible?

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## ■ Solution

*The number of possible winning numbers is the combination of 6 objects selected from 48, that is,*

$$\binom{48}{6} = \frac{48!}{6!(48-6)!} = 12271512.$$



# Example

Since each of the 12271512 possible winning numbers are equally likely, the probability that a player who buys one ticket wins is  $\frac{1}{12271512}$ .

## Example

A hand of seven cards is dealt from a deck of 52 cards.

- (A) How many hands are possible?
- (B) How many hands that contain exactly 2 diamonds are possible?
- (C) How many hands with 2 diamonds and 3 spades are possible?

# Solution

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- (A) *The number of seven-card hands is the number of choosing 7 elements from a set of 52 which is  $\binom{52}{7}$ .*

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- (B) *A hand with 2 diamonds can be formed as follows. Imagine,*
  - *deal 2 diamonds from the 13 diamonds.*
  - *deal 5 cards from the remaining 39 cards.*

*The number of possible hands is,*

$$\binom{13}{2} \binom{39}{5} = 44909046.$$

# Solution

## Solution

(C) A hand with 2 diamonds and 3 spades can be formed as follows.

- deal 2 diamonds from the 13 diamonds.
- deal 3 spades from the 13 spades.
- deal 2 cards from the remaining 26 cards.

The number of possible hands is,

$$\binom{13}{2} \binom{13}{3} \binom{26}{2} = 7250100.$$

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## Example

A box contains 15 balls; 3 red, 5 blue and 7 green balls. 8 balls are selected from the box without replacement. What is the probability that the balls selected are 1 red , 4 blue and 3 green?

# Solution

- The number of ways to select 8 balls from 15 is,  
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- The number of ways to select 4 blue balls from 5 blue is,  
$$\binom{5}{4} = \frac{5!}{4!(5-4)!} = 5.$$

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- The number of ways to select 3 green balls from 7 green is,  
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- The number of ways to select 3 green balls from 7 green is,  
$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = 35.$$
- Then by the Fundamental Principle of Counting, the number of ways to select 1 red ball, 4 blue balls and 3 green is,  
$$\binom{3}{1} \binom{5}{4} \binom{7}{3} = 3 \cdot 5 \cdot 35 = 525.$$

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- Then by the Fundamental Principle of Counting, the number of ways to select 1 red ball, 4 blue balls and 3 green is,  
$$\binom{3}{1} \binom{5}{4} \binom{7}{3} = 3 \cdot 5 \cdot 35 = 525.$$
- The probability of selecting 1 red ball, 4 blue balls and 3 green balls is,  
$$\frac{\binom{3}{1} \binom{5}{4} \binom{7}{3}}{\binom{15}{8}} = \frac{525}{6435} = \frac{105}{1287}.$$

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- ..
- ..
- $\binom{n}{n}$  subsets with  $n$  elements.
- The total number of possible subsets of  $S$  is  
$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{r=0}^n \binom{n}{r}.$$

# Binomial formula

- Binomial Formula:

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} b^r a^{n-r}.$$

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- It follows that,

$$2^n = \sum_{r=0}^n \binom{n}{r}.$$

- Hence, if a set has  $n$  elements, it has  $2^n$  different subsets.

# Example

## ■ Example

A coin is flipped 5 times, and the number of heads and tails is counted.

(A) How many flips have exactly 2 heads?

(B) How many flips have at most 2 heads?

(C) How many flips have at least 2 heads?

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## Solution

*(A) If there are exactly 2 heads, there are exactly 3 tails, so the number of such flips is the number of permutations of 2 heads and 3 tails. That number is  $\frac{5!}{2!3!} = 10$ .*

- (B) At most 2 heads means that one of the following is true:
  - no heads
  - exactly 1 head
  - exactly 2 heads
  - Hence  $\frac{5!}{0!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} = 1 + 5 + 10 = 16$  flips have at most 2 heads.

- (B) At most 2 heads means that one of the following is true:
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  - Hence  $\frac{5!}{0!5!} + \frac{5!}{1!4!} + \frac{5!}{2!3!} = 1 + 5 + 10 = 16$  flips have at most 2 heads.
- (C) At least 2 heads means that one of the following is true:
  - exactly 2 heads
  - exactly 3 heads
  - exactly 4 heads
  - exactly 5 heads
  - Hence  $\frac{5!}{2!3!} + \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!0!} = 10 + 10 + 5 + 1 = 26$  flips have at least 2 heads.

# Example

## Example

A coin is flipped  $n$  times.

- Total number of outcomes is the number of sequences of length  $n$  which is  $2^n$ .
- The number of sequences with  $r$  heads, is the number of subsets of size  $r$  of a set with  $n$  elements, which is  $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ .
- The probability of  $r$  heads in a sequence of  $n$  coin tosses is

$$\frac{\binom{n}{r}}{2^n}.$$

# Summary

	Order is Important	Order is Not Important
With Replacement	$n^r$	not covered
Without Replacement	${}_n P_r = \frac{n!}{(n-r)!}$	${}_n C_r = \binom{n}{r}$

## Example

Suppose a box contains 30 balls, of which 12 is white and 18 is black. Pick 4 balls without replacement. The total possible outcomes are  $\binom{30}{4}$ . Let  $k$  be the number of white balls chosen. Then we have chosen  $k$  white balls out of 12 and  $4 - k$  black balls out of 18. Then by the Fundamental Theorem of Counting, there are  $\binom{12}{k} \cdot \binom{18}{4-k}$  possible outcomes. Thus, the probability of choosing  $k$  white balls is then

$$\frac{\binom{12}{k} \cdot \binom{18}{4-k}}{\binom{30}{4}}.$$

## Example

```
> k <- -c(0 : 4)
> P <- -(choose(12, k) * choose(18, 4 - k))/(choose(30, 4))
> data.frame(k, P)
  k  P
1  0 0.11165846
2  1 0.35730706
3  2 0.36847291
4  3 0.14449918
5  4 0.01806240
```

# Birthday problem, chapter 2.4

## Example

Suppose  $k$  people are in a room

(A) what is the probability that no two of them have the same birthday?

(B) How many people must be in the room for the probability that at least two of the  $k$  people have the same birthday to be greater than  $\frac{1}{2}$ .

# Birthday problem, Solution

## Solution

(A) Since there are  $k$  people in the room and each has 365 possibilities for a birthday, there are  $365^k$  possible outcomes. The number of possible outcomes for which no 2 have the same birthday is  $(365)(364)\dots(365 - k + 1)$ . Thus, the probability, under the assumption that each outcome is equally likely, that no 2 people in the room have the same birthday is,  $\frac{(365)(364)\dots(365 - k + 1)}{365^k}$ .

(B) The probability that at least two people of the  $k$  people share a birthday is  $1 - \frac{(365)(364)\dots(365 - k + 1)}{365^k}$ . Note that we can write  $\frac{(365)(364)\dots(365 - k + 1)}{365^k} = \frac{(365)(364)\dots(365 - (k - 1) + 1)}{365^{k-1}} \frac{(365 - k + 1)}{365}$ . Using  $R$  (see other slide), we obtained that when  $k = 23$ , there is a 50% chance that at least two of the people in the room share a birthday. In a class of 38 students, there is a 86% chance that at least two people share a birthday.