Basic concepts, chapter 1.1

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Randomness

Example

Suppose we want to simulate 10 flips of a coin. To simulate a flip of a coin in R use the **sample** command. This is the command of sampling without replacement. To sample with replacement in R, we will call for the command **Replace = T**

Let x be a vector. The command: sample(x,n,replace=T) randomly choose n numbers from the vector x independently from each other and with replacement.

Simulate flips of a coin in R

Example

To simulate 10 flips of a coin in R:

Let 1 be associated with head and 0 with tail.

$> x < -c(0,1)$

$> x$

[1] 0 1

$>$ data$<$-sample(x,10,replace $=$ T)

$>$ data

[1] 1 1 0 0 1 0 1 0 1 1

These numbers corresponds to:

H H T T H T H T H H

From these 10 simulated flips, the estimated probability of heads is $6/10 = 0.6$

Example

To simulate 50 flips of a coin in R:

```
> x < -c(0,1)
> x
[1] 0 1
> data<-sample(x,50,replace = T)
> data
[1] 1 0 1 0 0 1 0 0 0 0 1 1 0 0 0 1 1 1 0 0 1 1 1 1 0 0 1 1 1 0 1 0 1
1 0 1 0 0
[39] 1 0 0 1 1 1 0 0 1 0 1 1
> sum(data)
[1] 26
```

From these 50 simulated flips, the estimated probability of heads is $26/50 = 0.52$

The limiting relative frequency of number of heads as the number of coin flips get large is 0.5.

Sample space

Definition

A sample space S is a set that includes all possible outcomes for a random experient listed in a mutually exclusive and exhaustive way.

Mutually exclusive: Outcomes do not overlap.
Exhaustive way: set contains all possible outcomes.

Sample space

Example

Suppose a six-sided die is rolled and the number on the upper face is observed.

$S^* = \{1, 2, 3, 4, 5, 6, \textit{even}, \textit{odd}\}$ is not a sample space since it is not mutually exclusive. For example 2 and even overlap.
$S^* = \{1, 2, 3, 4, 6\}$ is not a sample space since it is missing 5.

Sample spaces:
$S_1 = \{1, 2, 3, 4, 5, 6\}$
$S_2 = \{\textit{even}, \textit{odd}\}$

Events

We are interested in a particular outcome or a set of outcomes. These are the events of interest.

Definition

An event is any subset of a sample space.

Example

A is the event of "rolling a 6". $A = \{6\}$
B is the event of "rolling a number greater than 3". $B = \{4, 5, 6\}$
C is the event of "observing an odd number". $C = \{1, 3, 5\}$

$\emptyset$ is the empty set or the nullset. It contains no outcomes. $\emptyset$ is a subset of every set and is therefore an event.
The sample space, S, is an event in every experiment.

Events

Example

D is the event of "observing a seven" in rolling a die. $D = \emptyset$.

We say that an event, A, has occurred if the outcome of an experiment is contained in the event A.

Example

Suppose we roll a die. Let A be the event of "observing a 4".
Let B be the event of "observing a number smaller than 4".
Let C be the event of "observing an even number".
Then $A = \{4\}$.
$B = \{1, 2, 3\}$
$C = \{2, 4, 6\}$
Suppose the die shows a 4 on the upper face. Then the event A and C have occurred.

If a set has n elements, it has 2^n possible subsets.
Thus, if a sample space has n outcomes, it has 2^n possible events.

Example

Roll a dice. There are 6 outcomes and $2^6 = 64$ possible events.

Definition of Probability

Definition

A probability, P, is a function that assigns a real value to every event A so that the following axioms hold:

- $P(A) \geq 0$
- $P(S) = 1$
- If A_1, A_2,..., is a sequence of mutually exclusive events, (that is a sequence in which $A_i \cap A_j = \emptyset$ for any $i \neq j$), then $P\big(\cup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} P(A_i)$.

Probability rules

If follows that

- $P(\emptyset) = 0$
- $0 \leq P(A) \leq 1$
- If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
- $P(A') = 1 - P(A)$
- If $A \subset B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Equally likely outcomes

- If S is a finite sample space and A is an event, then if each outcome is equally likely, that is they have the same probability of occuring, the probability of A is $P(A) = \frac{N(A)}{N(S)}$, where $N(A)$ is the number of elements in A.

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- **Example**

  Roll a symmetric dice. The probability of obtaining, i, where $i = 1, 2, 3, 4, 5, 6$ is
  $P(\{i\}) = \frac{1}{6}$.

Equally likely outcomes

- If $A \cap B = \emptyset$, then $N(A \cup B) = N(A) + N(B)$ and hence $\frac{N(A \cup B)}{N(S)} = \frac{N(A)}{N(S)} + \frac{N(B)}{N(S)}$ or $P(A \cup B) = P(A) + P(B)$.

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- **Example**

  Roll a dice.

  - (a) The probability of obtaining a "one" or a "three" is $P(\{1\} \cup \{3\}) = P(\{1\}) + P(\{3\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.
  - (b) The probability of obtaining an odd number is: $P(Odd) = P(\{1\} \cup \{3\} \cup \{5\}) = P(\{1\}) + P(\{3\}) + P(\{5\}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$.

Example

Toss a coin 4 times. Let A be the event of "obtaining exactly 3 heads" Let B be the event of "obtaining exactly 4 heads".
(a) Find $P(A)$ and $P(B)$
(b) Find P(at least 3 heads)
(c) Find P(fewer than 3 heads)

Example

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Solution

(a)
$A=\{HHHT, HHTH, HTHH, THHH\}$
$B=\{HHHH\}$
$N(S)=16$, $N(A)=4$ *and* $N(B)=1$
$P(A)=\frac{4}{16}=\frac{1}{4}$ *and* $P(B)=\frac{1}{16}$.

Solutions continue

Solution

- (b) *P(at least 3 heads)*= *P(exactly 3 heads)*+*P(exactly 4 heads)*= $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$.

Solutions continue

Solution

- (b) P(at least 3 heads)= P(exactly 3 heads)+P(exactly 4 heads)= $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$.
- (c) P(fewer than 3 heads)= 1−P(at least 3 heads)=$1 - \frac{5}{16} = \frac{11}{16}$.

Example

If $P(A) = 0.3$, $P(B) = 0.4$, and $P(A \cap B) = 0.1$, find
(A)$P(A \cup B)$
(B)$P(A \cap B')$
(C)$P(A' \cup B')$

Example

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Solution

- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.1 = 0.6$
- $P(A \cap B') = P(A) - P(B) = 0.3 - 0.1 = 0.2$

Example

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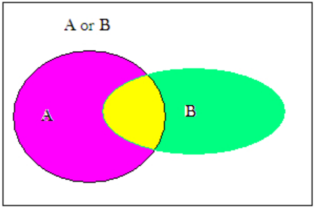
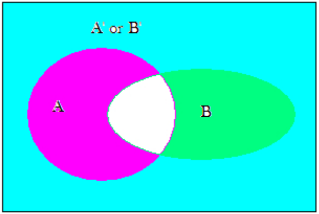
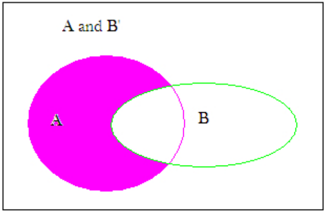
Solution

- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.1 = 0.6$
- $P(A \cap B') = P(A) - P(B) = 0.3 - 0.1 = 0.2$
- $P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B) = 1 - 0.1 = 0.9$

The following venn diagrams illustrates the previous problem and shows $(A \cup B)$, $(A \cap B')$, and $(A' \cup B')$.







Example

In a class of 60 students, 13 could not play an instrument, 17 are playing sport, and 10 could play an instrument and are playing sport. A student is randomly selected from this class. Find the probability that the selected student:
(a) Can play an instrument
(b) Cannot play an instrument and are playing sports.
(c) Can either play an instrument or are playing sports but not both.

Solution is given in class.