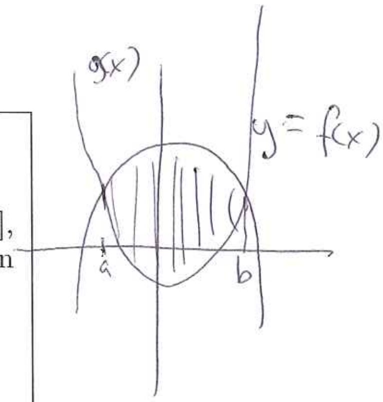
7.5 The Area Between Two Curves

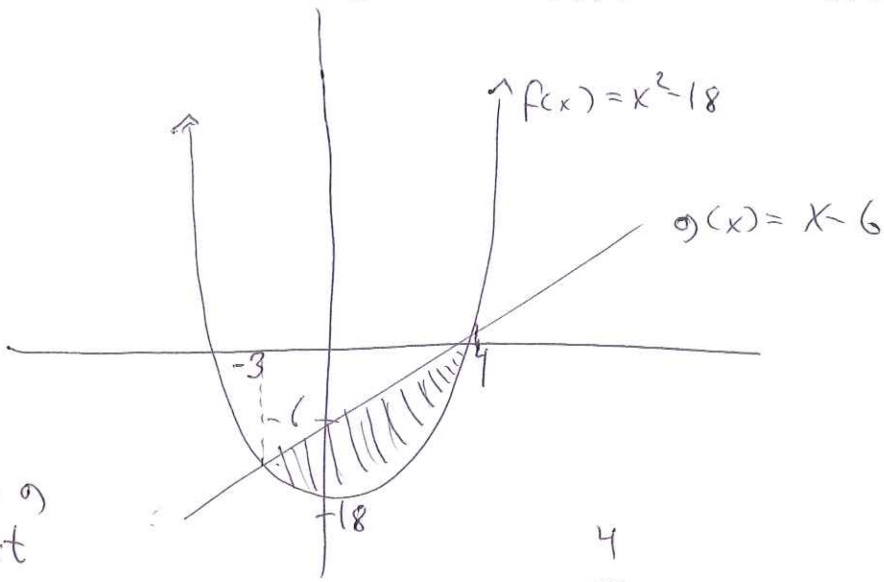


Area between two curves

Let f and g be continuous functions and $f(x) \geq g(x)$ on the interval $[a, b]$, then the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b [f(x) - g(x)]\, dx.$$

Example. Find the area of the region bounded by $f(x) = x^2 - 18$ and $g(x) = x - 6$.



1. Find where ~~Find~~ f and g intersect

$f(x) = g(x)$

$x^2 - 18 = x - 6$

$x^2 - x - 12 = 0$

$(x+3)(x-4) = 0$

$x + 3 = 0 \qquad x - 4 = 0$

$x = -3 \qquad x = 4$

$$A = \int_{-3}^{4} (g(x) - f(x))\, dx$$

$$= \int_{-3}^{4} \left(x - 6 - (x^2 - 18) \right) dx$$

$$= \int_{-3}^{4} (x + 12 - x^2)\, dx$$

$$= \left. \frac{x^2}{2} + 12x - \frac{x^3}{3} \right|_{-3}^{4} = \frac{4^2}{2} + 12 \cdot 4 - \frac{4^3}{3} - \left(\frac{(-3)^2}{2} + 12 \cdot (-3) - \frac{(-3)^3}{3} \right)$$

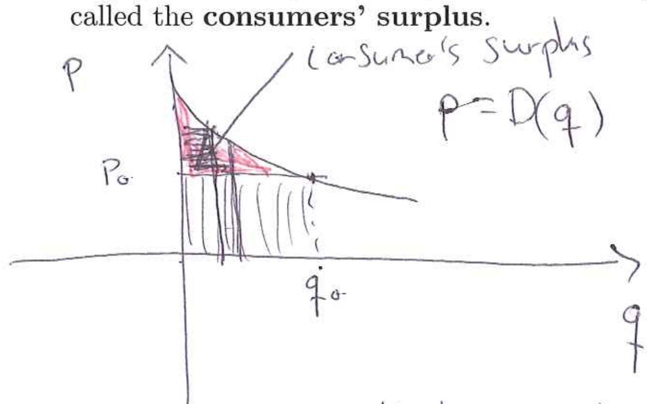
$$= 8 + 48 - \frac{64}{3} - \frac{9}{2} + 36 - 9 = \frac{343}{6}$$

Consumers' Surplus

The market determines the price at which a product is sold. Recall that the point of intersection of the demand curve and the supply curve for a product gives the equilibrium price and the equilibrium quantity.

At the equilibrium price, the quantity that the consumers will demand of a product is the same as the quantity that the manufacturers want to sell.

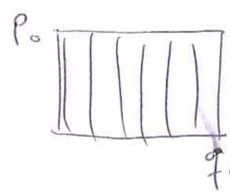
Some consumers, however, would be willing to spend more for an item than the equilibrium price. The total of the differences between the higher prices that individuals would be **willing** to pay and the equilibrium price of the item, is a saving realized by those individuals and is called the **consumers' surplus**.



p_0 = equilibrium price

q_0 = equilibrium quantity

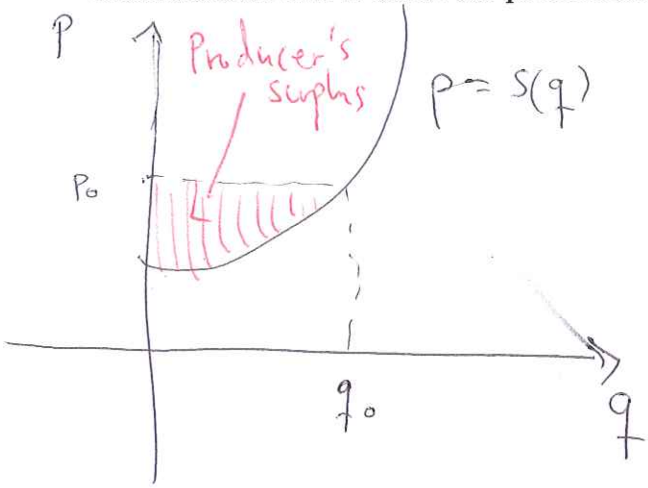
Suppose first that everybody is willing to pay exactly p_0. Then the total amount everybody is willing to pay is $p_0 \cdot q_0$



The total amount consumers are willing to ~~pay~~ spend for q_0 items is the area under the demand curve. The shaded area is the difference between what consumers would be willing to pay and what they actually pay. This is called the consumer's surplus.

> **Consumers' Surplus**
> If $D(q)$ is a demand function with equilibrium price p_0 and equilibrium demand q_0, then
> $$\textbf{Consumers surplus} = \int_0^{q_0} [D(q) - p_0]\, dq.$$

If some manufacturers would be willing to supply a product a at price **lower** than the equilibrium price p_0, the total of the differences between the equilibrium price and the lower prices at which the manufacturers would sell the product is considered added income for the manufacturers and is called the **producers' surplus**.



The total area under the line $p = p_0$ is the amount actually realized. The area under the supply curve from $q = 0$ to $q = q_0$ is the minimum total amount manufactures are willing to realize from the sale of q_0 items. The difference between these two areas (red shaded area) is called the producers' surplus.

Producers' Surplus
If $S(q)$ is a supply function with equilibrium price p_0 and equilibrium supply q_0, then

$$\textbf{Producers' surplus} = \int_0^{q_0} [p_0 - S(q)]\, dq.$$

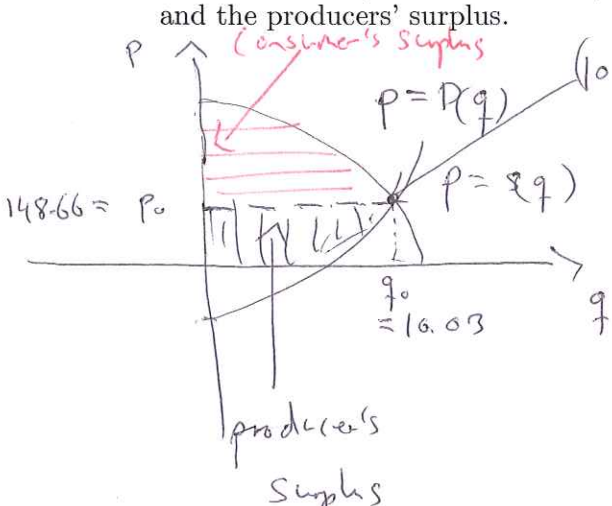
Example. Suppose the price (in dollars per ton) for a certain product is

$$D(q) = 300 - e^{0.5q},$$

when the demand for the product is q tons. Also, suppose the function

$$S(q) = e^{0.5q} - 2$$

gives the price (in dollars per ton) when the supply is q tons. Find the consumers' surplus and the producers' surplus.



$$D(q) = S(q)$$

$$300 - e^{0.5q} = e^{0.5q} - 2$$

$$302 = 2e^{0.5q}$$

$$\frac{302}{2} = e^{0.5q}$$

$$151 = e^{0.5q} \qquad \ln 151 = \ln e^{0.5q}$$

$$\ln(151) = 0.5q$$

$$q = \frac{\ln(151)}{0.5} \approx 10.03$$

$q_0 = 10.03$ Equilibrium quantity

$$p = D(10.03) = 300 - e^{0.5 \cdot 10.03} \approx 148.66$$

Equilibrium price: $p_0 = 148.66$

Consumer's surplus

$$= \int_0^{10.03} (D(q) - p_0)\, dq$$

$$= \int_0^{10.03} (300 - e^{0.5q} - 148.66)\, dq$$

$$= \int_0^{10.03} (150.66 - e^{0.5q})\, dq$$

$$= 150.66q - \frac{1}{0.5} e^{0.5q} \Big|_0^{10.03}$$

$$= 150.66 \cdot (10.03) - \frac{1}{0.5} e^{0.5 \cdot 10.03} - \left(-\frac{1}{0.5} e^{0.5 \cdot 0}\right)$$

$$= 150.66 \cdot (10.03) - \frac{1}{0.5} e^{0.5 \cdot (10.03)} + \frac{1}{0.5} e^{0} = \underline{1211.81 \text{ dollar}}$$

Producers' surplus

$$= \int_0^{10.03} 148.66 - (e^{0.5q} - 2)\, dq$$

$$= \int_0^{10.03} 150.66 - e^{0.5q}\, dq = \underline{\underline{1211.81 \text{ dollar}}}$$

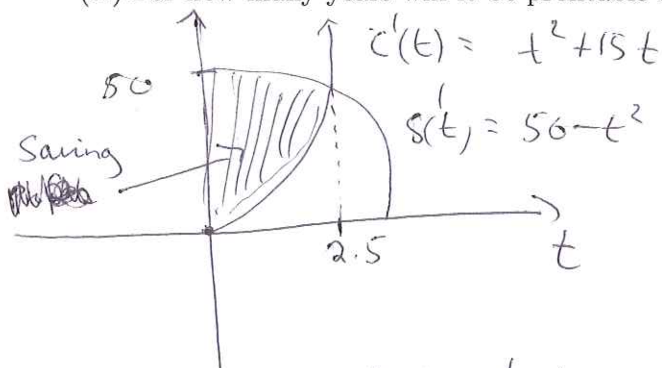
Example. Suppose a company wants to introduce a new machine that will produce a rate of annual savings (in dollars) given by

$$S'(t) = 50 - t^2,$$

where t is the number of years of operation of the machine, while producing a rate of annual costs (in dollars) of

$$C'(t) = t^2 + 15t$$

(A) For how many years will it be profitable to use this new machine?



$S'(t) = C'(t)$

$50 - t^2 = t^2 + 15t$

$50 - 2t^2 - 15t = 0$

$(2t-5)(-10-t) = 0$

$2t - 5 = 0$ $\quad$ $-10 - t = 0$

$t = 2.5$ $\quad$ $t = \cancel{-10}$

It will be profitable to use this machine for $\underline{2.5 \text{ years}}$.

(B) What are the net total saving over the entire period of use of the machine?

Total net saving over the first 2.5 years

$$= \int_0^{2.5} (S'(t) - C'(t))\,dt = \int_0^{2.5} \left[50 - t^2 - (t^2 + 15t)\right] dt$$

$$= \int_0^{2.5} \left[-2t^2 + 50 - 15t\right] dt = -\frac{2t^3}{3} + 50t - \frac{15t^2}{2} \Bigg|_0^{2.5}$$

$$= -\frac{2}{3}(2.5)^3 + 50(2.5) - \frac{15}{2}\cdot(2.5)^2 - 0$$

$$\approx 67.71 \text{ dollars}$$