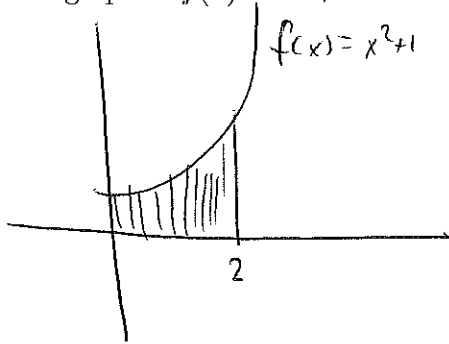


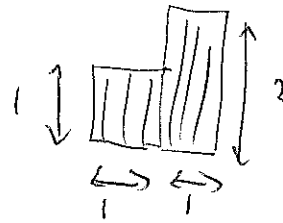
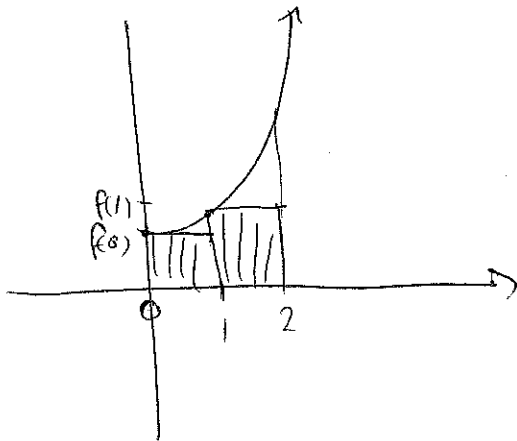
7.3 Area and the Definite Integral

In this chapter, we will approximate areas under curves.

Example. Consider the region bounded by the y-axis, the x-axis, the line $x = 2$, and the graph of $f(x) = x^2 + 1$.



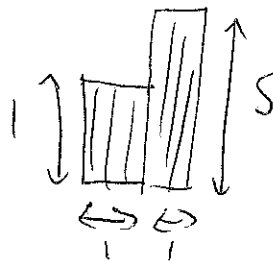
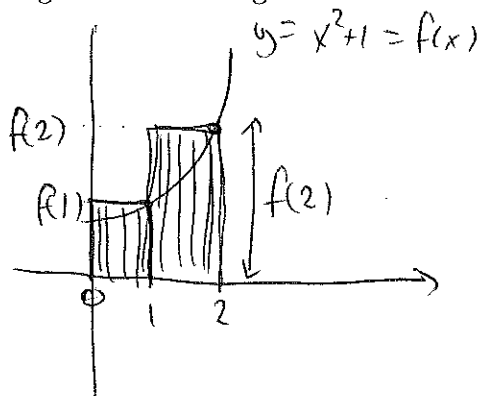
(A) Approximate the area of the region using two rectangles. Determine the height of the rectangle by the value of the function at the **left** endpoint.



Area of rectangles:

Left Sum: $f(0) \cdot 1 + f(1) \cdot 1 = 1 \cdot 1 + 1 \cdot 2 = \underline{\underline{3}}$ Square units

(B) Repeat part (A) using the value of the function at the **right** endpoint to determine the height of the rectangle.



Area of rectangles:

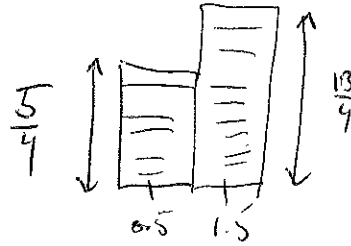
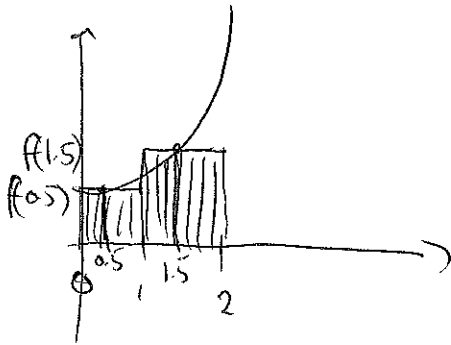
Right
Sum: $f(1) \cdot 1 + f(2) \cdot 1 = 2 \cdot 1 + 5 \cdot 1 = \underline{\underline{7}}$ Square units

(C) Find the average value of the two values in part (A) and part (B).

$$\frac{3+7}{2} = \underline{\underline{5}} \text{ Square units}$$

Trapezoidal rule: Average value \nearrow

(D) Repeat part (A) using the value of the function at the **midpoint** of each interval to determine the height of the rectangle.



Area of rectangles:

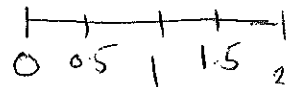
$$\begin{aligned} f(0.5) \cdot 1 + f(1.5) \cdot 1 &= \frac{5}{4} \cdot 1 + \frac{13}{4} \cdot 1 \\ &= \frac{9}{2} = \underline{\underline{4.5}} \text{ Square units.} \end{aligned}$$

(E) We can improve the accuracy of the previous approximations by increasing the number of rectangles. Repeat part (A)-(D) using four rectangles.

Length of rectangle: 0.5

Left Sum:

$$\begin{aligned} A &= f(0) \cdot 0.5 + f(0.5) \cdot 0.5 + f(1) \cdot 0.5 + f(1.5) \cdot 0.5 \\ &= 1 \cdot 0.5 + \frac{5}{4} \cdot 0.5 + 2 \cdot 0.5 + \frac{13}{4} \cdot 0.5 = \underline{\underline{3.75}} \end{aligned}$$



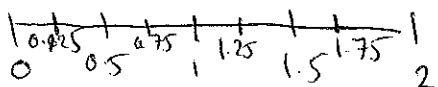
Right Sum: $A = f(0.5) \cdot 0.5 + f(1) \cdot 0.5 + f(1.5) \cdot 0.5 + f(2) \cdot 0.5$

$$= 0.5 \cdot \left(\frac{5}{4} + 2 + \frac{13}{4} + 5 \right) = \underline{\underline{5.75}}$$

Average: $\frac{3.75 + 5.75}{2} = \underline{\underline{4.75}}$

Midpoint:

$$\begin{aligned} A &= f(0.25) \cdot 0.5 + f(0.75) \cdot 0.5 + f(1.25) \cdot 0.5 \\ &\quad + f(1.75) \cdot 0.5 = \underline{\underline{4.625}} \end{aligned}$$



n	Left Sum	Right Sum	Trapezoidal	Midpoint
2	3	7	5	4.75
4	3.75	5.75	4.75	4.625
⋮				
1000	4.6627	4.6707	4.67	

Thus,

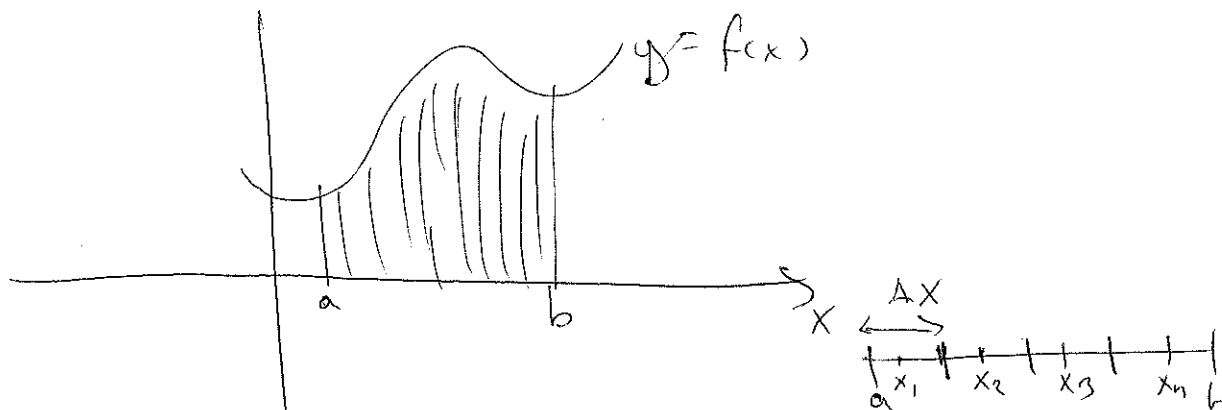
$$\int_0^2 (x^2 + 1) dx \approx \underline{\underline{4.67}}$$

Divide the interval from $x = 0$ to $x = 2$ into n equal parts. Each of these n intervals has width

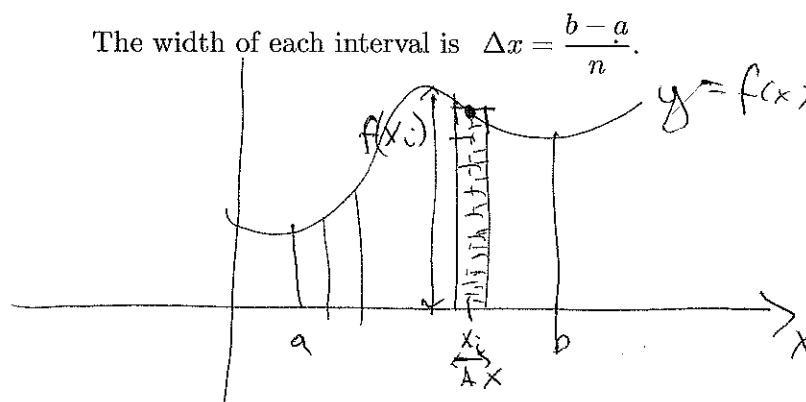
$$\frac{2 - 0}{n} = \frac{2}{n}$$

and height determined by the function value at the left side of the rectangle, or the right side, or the midpoint.

Area under the curve We will now find a method to find the area bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. Assume first $f(x) \geq 0$.



1. Divide the interval from a to b into n pieces of equal width. Each of these n pieces are used as the base of a rectangle.
2. Let x_1 be an arbitrary point in the first interval, x_2 be an arbitrary point in the second interval, and so forth, up to the n^{th} interval.
- 3.



4. The i^{th} rectangle has width Δx and length given by the height $f(x_i)$.
The area of the i^{th} rectangle = $f(x_i) \cdot \Delta x$.

5. The total area under the curve is approximated by the sum of the areas of all n of the rectangles.

$$\text{Area of all } n \text{ rectangles} = \sum_{i=1}^n f(x_i) \Delta x = f(x_1) \cdot \Delta x + f(x_2) \Delta x + \dots + f(x_n) \Delta x$$

6. The exact area is defined to be the limit of this sum (if the limit exists) as the number of rectangles increases without bound:

$$\text{Exact area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x.$$

The definite Integral

If f is defined on the interval $[a, b]$, the **definite integral** of f from a to b is given by

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

provided the limit exists, where $\Delta x = \frac{b-a}{n}$, and x_i is any value of x in the i^{th} interval.

The a below the integral sign is called the **lower limit** of integration and b is called the **upper limit** of integration.

Approximation of the definite Integral

The definite integral can be approximated by

$$\sum_{i=1}^n f(x_i) \Delta x.$$

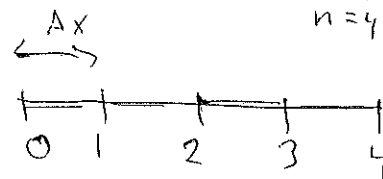
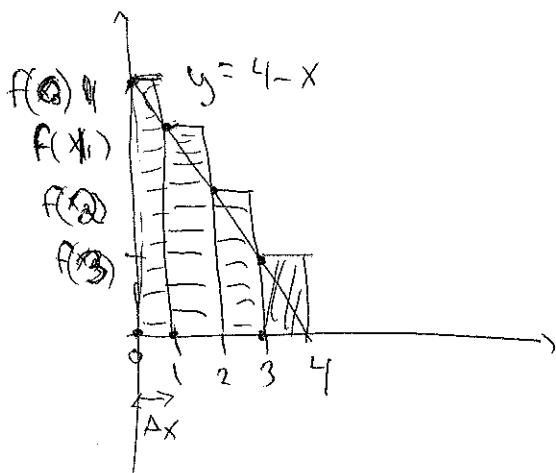
If $f(x) \geq 0$ on the interval $[a, b]$, the definite integral gives the area under the curve between $x = a$ and $x = b$.

1. In the midpoint rule, x_i is the midpoint of the i^{th} interval.
2. For the left sum, x_i is the left endpoint of the i^{th} interval.
3. For the right sum, x_i is the right endpoint of the i^{th} interval.

Example. Approximate

$$\int_0^4 (4-x) dx \approx \sum_{i=1}^4 f(x_i) \Delta x$$

by using four rectangles and left endpoints (left sums).



$$\begin{aligned} a &= 0 \\ b &= 4 \\ n &= 4 \end{aligned}$$

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

$$\begin{aligned} \int_0^4 (4-x) dx &= f(x_0) \Delta x + f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x \\ &= f(0) \Delta x + f(1) \Delta x + f(2) \Delta x + f(3) \Delta x \end{aligned}$$

Total Change

Example. Suppose $f(x) = x^2 + 10$ gives the marginal cost of some item at a particular x -value. Find the **total change** in the cost function as x changes from 2 to 4.

$$f(2) = 2^2 + 10 = 14 \text{ is the rate of change in cost}$$

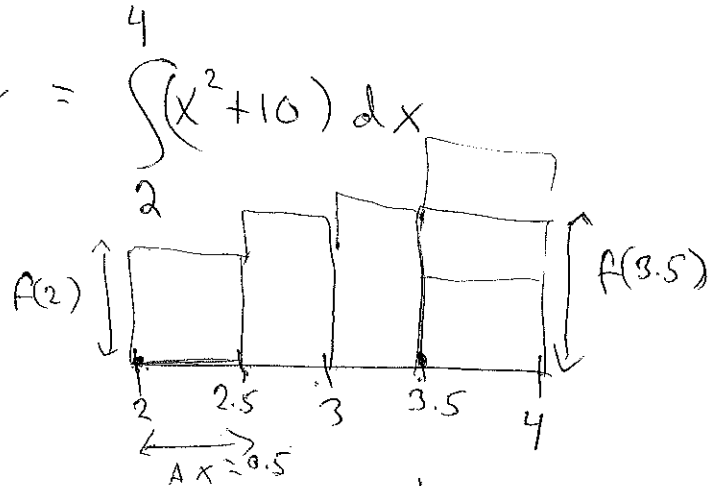
when $x=2$.

$$\text{Total change in cost} = \int_2^4 f(x) dx = \int_2^4 (x^2 + 10) dx$$

Use
TRAPEZOIDAL

rule
use 4 rectangles

SUM
LEFT RULE:



$$\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = 0.5$$

$$f(2) \cdot 0.5 + f(2.5) \cdot 0.5 + f(3) \cdot 0.5 + f(3.5) \cdot 0.5 =$$

$$0.5 (14 + (2.5)^2 + 10 + 3^2 + 10 + (3.5)^2 + 10) = 35.75$$

Total change in the cost

The area of the region under the graph of the marginal cost function $f(x)$ that is above the x -axis and between $x = a$ and $x = b$ gives the **net total change in the cost** as x goes from a to b .

SUM

$$\text{RIGHT RULE: } f(2.5) \cdot 0.5 + f(3) \cdot 0.5 + f(3.5) \cdot 0.5 + f(4) \cdot 0.5$$

$$= 0.5 \cdot ((2.5)^2 + 10 + 3^2 + 10 + (3.5)^2 + 10 + 4^2 + 10)$$

$$= 0.5 (6.25 +$$

$$= 41.75$$

$$\text{TRAPEZOIDAL RULE} = \frac{\text{LEFT SUM} + \text{RIGHT SUM}}{2} = \frac{35.75 + 41.75}{2} = 38.75 \text{ Dollar}$$

Total change in $F(x)$

$$F'(x) = f(x)$$

If $f(x)$ gives the rate of change of $F(x)$ for x in $[a, b]$, then the **total change** in $F(x)$ as x goes from a to b is given by

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \int_a^b f(x) dx.$$