7.3 Area and the Definite Integral

In this chapter, we will approximate areas under curves.

Example. Consider the region bounded by the y-axis, the x-axis, the line $x = 2$, and the graph of $f(x) = x^2 + 1$.

(A) Approximate the area of the region using two rectangles. Determine the height of the rectangle by the value of the function at the **left** endpoint.

(B) Repeat part (A) using the value of the function at the **right** endpoint to determine the height of the rectangle.

(C) Find the average value of the two values in part (A) and part (B).

(D) Repeat part (A) using the value of the function at the **midpoint** of each interval to determine the height of the rectangle.

(E) We can improve the accuracy of the previous approximations by increasing the number of rectangles. Repeat part (A)-(D) using four rectangles.

Divide the interval from $x = 0$ to $x = 2$ into n equal parts. Each of these n intervals has width

$$\frac{2-0}{n} = \frac{2}{n}$$

and height determined by the function value at the left side of the rectangle, or the right side, or the midpoint.

Area under the curve We will now find a method to find the area bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. Assume first $f(x) \geq 0$.

1. Divide the interval from a to b into n pieces of equal width. Each of these n pieces are used as the base of a rectangle.

2. Let x_1 be an arbitrary point in the first interval, x_2 be an arbitrary point in the second interval, and so forth, up to the n^{th} interval.

3.

$$\text{The width of each interval is} \quad \Delta x = \frac{b-a}{n}.$$

4. The i^{th} rectangle has width Δx and length given by the height $f(x_i)$.

$$\text{The area of the} \quad i^{th} \quad \text{rectangle} = f(x_i) \cdot \Delta x.$$

5. The total area under the curve is approximated by the sum of the areas of all n of the rectangles.

$$\text{Area of all n rectangles} \quad = \sum_{i=1}^{n} f(x_i)\, \Delta x.$$

6. The exact area is defined to be the limit of this sum (if the limit exists) as the number of rectangles increases without bound:

$$\text{Exact area} \quad = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\, \Delta x.$$

The definite Integral

If f is defined on the interval $[a, b]$, the **definite integral** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

provided the limit exists, where $\Delta x = \frac{b-a}{n}$, and x_i is any value of x in the i^{th} interval.
The a below the integral sign is called the **lower limit** of integration and b is called the **upper limit** of integration.

Approximation of the definite Integral

The definite integral can be approximated by

$$\sum_{i=1}^{n} f(x_i)\,\Delta x.$$

If $f(x) \geq 0$ on the interval $[a, b]$, the definite integral gives the area under the curve between $x = a$ and $x = b$.
1. In the midpoint rule, x_i is the midpoint of the i^{th} interval.
2. For the left sum, x_i is the left endpoint of the i^{th} interval.
3. For the right sum, x_i is the right endpoint of the i^{th} interval.

Example. Approximate

$$\int_0^4 (4 - x)\,dx$$

by using four rectangles and left endpoints (left sums).

Total Change

Example. Suppose $f(x) = x^2 + 10$ gives the marginal cost of some item at a particular x-value. Find the **total** change in the cost function as x changes from 2 to 4.

Total change in the cost

The area of the region under the graph of the marginal cost function $f(x)$ that is above the x-axis and between $x = a$ and $x = b$ gives the **net total change in the cost** as x goes from a to b.

Total change in $F(x)$

If $f(x)$ gives the rate of change of $F(x)$ for x in $[a, b]$, then the **total change** in $F(x)$ as x goes from a to b is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx.$$