7.2 Substitution

The differential of μ

If $u = f(x)$, the **differential** of u, written du is defined as

$$du = f'(x)dx.$$

$$\frac{du}{dx} = f'(x)$$

Example. If $u = x^2 + 3$, then $du =$ $2x\,dx$

$f(x)$ $\quad \frac{du}{dx} = 2x$

Integration by substitution

Example. Find $\int \underbrace{(x^2+3)^4 2x\,dx}_{f(x)}$

$u = x^2 + 3$

$du = 2x\,dx$

Substitute u for x^2+3 and du for $2x\,dx$ in the indefinite integral

Power rule $\quad n=4$

$$\int \underbrace{(x^2+3)}_{u}{}^4 \underbrace{2x\,dx}_{du} = \int u^4\,du = \frac{u^{4+1}}{4+1} + C$$

$u = x^2+3$

$$= \frac{(x^2+3)^5}{5} + C$$

$F(x)$

Check: $\frac{d}{dx}\left(\frac{(x^2+3)^5}{5} + C\right) = \frac{1}{\cancel{5}}\cancel{5}(x^2+3)^4 \cdot 2x = (x^2+3)^4 \cdot 2x$

$$u = 2x^4 - 1$$

$$du = 8x^3\,dx$$

Example. Find $\int 8x^3 (2x^4 - 1)^5 \, dx$

$$= \int (2x^4 - 1)^5 \, 8x^3 \, dx$$

$$= \int u^5 \, du = \frac{u^{5+1}}{5+1} + C = \frac{(2x^4 - 1)^6}{6} + C$$

$u = x^3 - 5$

$du = 3x^2\,dx$

Example. Find $\int x^2\sqrt{x^3-5}\,dx$

$\int x^2\sqrt{x^3-5}\,dx = \frac{3}{3}\cdot\int\sqrt{x^3-5}\;x^2\,dx$

$= \frac{1}{3}\int 3\cdot\sqrt{x^3-5}\;x^2\,dx = \frac{1}{3}\int\underbrace{\sqrt{x^3-5}}_{u}\;\underbrace{3x^2\,dx}_{du}$

Power rule

$= \frac{1}{3}\int\sqrt{u}\;du = \frac{1}{3}\int u^{\frac{1}{2}}\,du = \frac{1}{3}\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$= \frac{1}{3}\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3}\cdot\frac{2}{3}\,(x^3-5)^{\frac{3}{2}} + C$

$= \frac{2}{9}\,(x^3-5)^{\frac{3}{2}} + C$

How to choose u

1. u should equal some expression in the integral that, when replaced with u, tends to make the integral simpler.
2. u must be an expression whose derivative, disregarding any constant multiplier, such as the 4 in $4x^3$, is also present in the integral.

Example. Find $\dfrac{x+6}{(x^2+12x)^4}\,dx$

$$u = x^2 + 12x$$

$$du = (2x+12)\,dx$$

$$du = 2(x+6)\,dx$$

1. method:

$$\int \frac{x+6}{(x^2+12x)^4}\,dx = \frac{2}{2}\int \frac{x+6}{(x^2+12x)^4}\,dx = \frac{1}{2}\int \frac{\overbrace{2(x+6)\cdot dx}^{du}}{\underbrace{(x^2+12x)}_{u}{}^4}$$

$$= \frac{1}{2}\int \frac{du}{u^4} = \frac{1}{2}\int \frac{1}{u^4}\,du$$

$$= \frac{1}{2}\int u^{-4}\,du = \frac{1}{2}\,\frac{u^{-4+1}}{-4+1} + C$$

$$= \frac{1}{2}\,\frac{u^{-3}}{-3} + C = -\frac{1}{6}(x^2+12x)^{-3} + C$$

2. method:

$$u = x^2 + 12x$$

$$du = 2(x+6)\,dx$$

$$\frac{du}{2} = (x+6)\,dx$$

$$\int \frac{x+6}{(x^2+12x)^4}\,dx = \cancel{\int \frac{1}{u^4}\,\frac{du}{2}}$$

$$= \int \frac{1}{(x^2+12x)^4}\underbrace{(x+6)\cdot dx}_{\frac{du}{2}} = \int \frac{1}{u^4}\,\frac{du}{2} = \frac{1}{2}\int \frac{1}{u^4}\,du$$

Same as method 1 from here.

Recall that

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}.$$

The integral

$$\int \frac{f'(x)}{f(x)}\,dx$$

can be solved by letting u equal the expression in the denominator, $u = f(x)$, since $du = f'(x)dx$ is present in the numerator.

Example. Find $\displaystyle\int \frac{3x^2+2}{(x^3+2x)}\,dx$

$$\int \frac{f'(x)}{f(x)}\,dx$$

$$u = x^3 + 2x$$

$$du = (3x^2+2)\,dx$$

$$\int \frac{3x^2+2}{x^3+2x}\,dx$$

$$= \int \frac{du}{u} = \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|x^3+2x| + C$$

Recall that

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x).$$

To integrate

$$\int e^{f(x)} f'(x)\,dx$$

let $u = f(x)$.

Example. Find $\int x^3 e^{x^4}\,dx$

$u = x^4$

$du = 4x^3\,dx$

$$\int x^3 e^{x^4}\,dx$$

$$= \frac{4}{4}\int x^3 e^{x^4}\,dx = \frac{1}{4}\int 4x^3 e^{x^4}\,dx$$

$$= \frac{1}{4}\int e^{x^4}\underbrace{4x^3\,dx}_{du} = \frac{1}{4}\int e^u\,du$$

$$= \frac{1}{4}e^u + C$$

$$= \cancel{\frac{1}{4}e^{x^3+2x} + C}$$

$$= \frac{1}{4}e^{x^4} + C$$

Find

$$\int e^{x^5} \cdot 2x^4\,dx$$

$u = x^5$

$du = 5x^4\,dx$

$$= \frac{5}{5}\int e^{x^5} \cdot 2x^4\,dx$$

$$= \frac{1}{5}\cdot 2\int e^{x^5}\cdot\underbrace{5x^4\,dx}_{du} = \frac{2}{5}\int e^u\,du = \frac{2}{5}e^u + C$$

$$= \frac{2}{5}e^{x^5} + C$$

Substitution

Each of the following forms can be integrated using the substitution $u = f(x)$.

1. $\int [f(x)]^n f'(x)\, dx$, $n \neq -1$

Result: $\int u^n \, du = \frac{u^{n+1}}{n+1} + C = \frac{[f(x)]^{n+1}}{n+1} + C.$

2. $\int \frac{f'(x)}{f(x)}\, dx.$

Result: $\int \frac{1}{u}\, du = \ln|u| + C = \ln|f(x)| + C$

3. $\int e^{f(x)} f'(x)\, dx.$

Result: $\int e^u \, du = e^u + C = e^{f(x)} + C.$

Example. Find $\int x\sqrt{3-x}\,dx$

$u = 3 - x \Rightarrow x = 3 - u$

$du = -\,dx$

$$= \int (3-u)\sqrt{u}\,(-du)$$

$$= -\int (3-u)\sqrt{u}\,du$$

$$= -\int (3-u)\,u^{\frac{1}{2}}\,du = -\int \left(3u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$$

$$= -\left(3\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right) + C = -\left(3\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right) + C$$

$$= -\left(\cancel{3}\cdot\frac{2}{\cancel{3}}\,u^{\frac{3}{2}} - \frac{2}{5}\,u^{\frac{5}{2}}\right) + C = -2\,(3-x)^{\frac{3}{2}} + \frac{2}{5}\,(3-x)^{\frac{5}{2}} + C$$

Substitution Method

In general, for the types of problems we are concerned with, there are three cases. We choose u to be one of the following:

1. the quantity under a root or raised to a power;
2. the quantity in the denominator;
3. the exponent on e.

Remember that some integrands may need to be rearranged to fit one of these cases.

Example. A company incurs debt at a rate of

$$D'(t) = 80(t+5)\sqrt{t^2+10t}$$

dollars per year, where t is the amount of time (in years) since the company began. By the third year the company had accumulated $12,000 in dept.
Find the total debt function.

$D(t) = \int D'(t)\,dt = \int 80(t+5)\sqrt{t^2+10t}\,dt$

$= 80\int \sqrt{t^2+10t}\,(t+5)\,dt$

$u = t^2+10t$

$du = 2t+10\,dt$

$= 2(t+5)dt$

$\frac{du}{2} = (t+5)dt$

$= 80\int \sqrt{u}\,\frac{1}{2}\,du$

$= 40\int u^{\frac{1}{2}}\,du = 40\,\frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{80}{3}(t^2+10t)^{\frac{3}{2}} + C$

$D(3) = 12{,}000$

$12000 = \frac{80}{3}(3^2+10\cdot 3)^{\frac{3}{2}} + C$

$450 \approx 243.56 + C$

$C = 450 - 243.56 = 206.44$

$D(t) = \frac{80}{3}(t^2+10t)^{\frac{3}{2}} + 206.44$