7.1 Antiderivatives

The reverse of finding a derivative is known as **antidifferentiation**. The antiderivative is defined as follows:

> **Antiderivative:**
>
> If $F'(x) = f(x)$, then $F(x)$ is an **antiderivative** of $f(x)$.

$F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Example. If $F(x) = x^2$, then $F'(x) = 2x$, making $F(x) = x^2$ an antiderivative of $f(x) = 2x$.

Example. Find an antiderivative of $f(x) = 4x^3$.

We know that the derivative of x^n is $n \cdot x^{n-1}$

We want to find a function $F(x)$ whose derivative is $f(x)$.

If nx^{n-1} is $4x^3$ then

$n-1=3$

So $n=4$, so $F(x) = x^4$

$F'(x) = 4x^3$

Example. Find three antiderivatives of $f(x) = 2x$.

~~Find an ant:~~

There is a family of functions having $2x$ as an antiderivative

$F(x) = x^2$ since $F'(x) = 2x$

$G(x) = x^2 + 1$ since $G'(x) = 2x$

$H(x) = x^2 - 1$ since $H'(x) = 2x$

$F(x) - G(x)$

$= x^2 - (x^2+1)$

$= -1 = C$

> If $F(x)$ and $G(x)$ are both antiderivatives of a function $f(x)$ on an interval, then there is a constant C such that
> $$F(x) - G(x) = C.$$

Find an antiderivative of $f(x) = x^2$

$\frac{d}{dx}(x^3) = 3x^2$

$\frac{1}{3}\frac{d}{dx}(x^3) = x^2$

$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$

So $\boxed{F(x) = \frac{x^3}{3}}$

Indefinite Integral

If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C$$

for any real number C.

The symbol $\int$ is the **integral sign**

$f(x)$ is the **integrand**

$\int f(x)\,dx$ is called an **indefinite integral**

The dx in the indefinite integral indicates that $\int f(x)\,dx$ is the integral of $f(x)$ with respect to the variable x. If $f(x) = 2x$, then $F(x) = x^2$ since $F'(x) = 2x$

Integrate $\int 2x\,dx = F(x) + C = \underline{\underline{x^2 + C}}$

Example. Find $\int 2ax\,dx = \int a(2x)\,dx = a \cdot \int (2x)\,dx$

X is the variable

$= \underline{\underline{a \cdot x^2 + C}}$

Example. Find $\int 2ax\,da = \int x(2a)\,da$

If $f(a) = 2a$

$F(a) = a^2$

a is the variable

$= x \int (2a)\,da$

$= \underline{\underline{x \cdot a^2 + C}}$

$\int f(x)\,dx = F(x) + C$

anti derivative

Power Rule:
For any real number ~~$n \neq 1$,~~ $n \neq -1$

$$\int \overbrace{x^n}^{f(x)}\,dx = \overbrace{\frac{x^{n+1}}{n+1}}^{F(x)} + C.$$

$F'(x) = f(x)$

$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}\frac{d}{dx}\left(x^{n+1}\right)$

$= \frac{1}{\cancel{n+1}}\cancel{(n+1)} \cdot x^{n+1-1}$

$= x^n$

Example. Find each indefinite integral below:

(A) $\int t^4\,dt$ $\quad n = 4$

$= \frac{t^{4+1}}{4+1} + C = \frac{t^5}{5} + C$

check: $\frac{d}{dt}\left(\frac{t^5}{5} + C\right) = \frac{1}{\cancel{5}} \cdot \cancel{5}t^4 + 0$

$= t^4$

(B) $\int \frac{1}{t^3}\,dt = \int \underbrace{t^{-3}}_{f}\,dt = \underbrace{\frac{t^{-3+1}}{-3+1}}_{F} + C = \frac{t^{-2}}{-2} + C$

$n = -3$

$F \quad + C$

(C) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$

$n = \frac{1}{2}$

$= \frac{2}{3}x^{\frac{3}{2}} + C$

(D) $\int dx = \int 1 \cdot dx = \int x^0\,dx$

$n = 0$

$= \frac{x^{0+1}}{0+1} + C = x + C$

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check $\frac{d}{dx}(x) = 1$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Constant Multiple Rule and Sum or Difference Rule:

If all indicated integral exist,

$$\int k \cdot f(x)\,dx = k \int f(x)\,dx \quad \text{for any real number} \quad k$$

and

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Example. Find each integral

(A) $\int 3x^2\,dx$ $= 3\int x^2\,dx = 3\left(\frac{x^{2+1}}{2+1}\right) + C$

$k = 3$

$n = 2$

$= \cancel{3}\,\frac{x^3}{\cancel{3}} + C = \underline{x^3 + C}$

check $\frac{d}{dx}(x^3 + C) = 3x^2$

(B) $\int \frac{10}{z^4}\,dz$

$= \int 10 \cdot z^{-4}\,dz = 10\int z^{-4}\,dz = 10 \cdot \left(\frac{z^{-4+1}}{-4+1}\right) + C$

$= 10 \cdot \frac{z^{-3}}{-3} + C$

$= -\frac{10}{3} z^{-3} + C$

(C) $\int (5t^7 + 3t - 4)\,dt$

$= \int 5t^7\,dt + \int 3t\,dt + \int -4\,dt$

$= 5\int t^7\,dt + 3\int t\,dt - 4\int dt$

$= 5 \cdot \frac{t^{7+1}}{7+1} + 3\,\frac{t^{1+1}}{1+1} - 4t + C$

$= \frac{5}{8} t^8 + \frac{3}{2} t^2 - 4t + C$

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(D) $\int \frac{t+2}{\sqrt{t}}\,dt$

$= \int \frac{t}{\sqrt{t}} + \frac{2}{\sqrt{t}}\,dt$

$= \int t \cdot t^{-\frac{1}{2}}\,dt + 2\int t^{-\frac{1}{2}}\,dt$

$$= \int t^{\frac{1}{2}}\,dt + 2\int t^{-\frac{1}{2}}\,dt$$

$$= \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2\,\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + 2\,\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{3}\,t^{\frac{3}{2}} + 2 \cdot \frac{2}{1} \cdot t^{\frac{1}{2}} + C$$

$$= \underline{\underline{\frac{2}{3}\,t^{\frac{3}{2}} + 4\,t^{\frac{1}{2}} + C}}$$

$$\frac{d}{dx} e^{kx} = ke^{kx}$$

Indefinite Integrals of Exponential Functions:

$\int e^x \, dx = e^x + C$

$\int e^{kx} \, dx = \frac{e^{kx}}{k} + C, \quad k \neq 0$

For $a > 0$, $a \neq 1$,

$\int a^x \, dx = \frac{a^x}{\ln a} + C$

$\int a^{kx} \, dx = \frac{a^{kx}}{k(\ln a)} + C, \quad k \neq 0.$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx}\left(\frac{e^{kx}}{k}\right) = \frac{1}{k} \frac{d}{dx} e^{kx}$$

$$= \frac{1}{\cancel{k}} e^{kx} \cdot \cancel{k} = e^{kx}$$

Example. Find each intergral

(A) $\int 5e^t \, dt$ $= 5 \int e^t \, dt = \underline{\underline{5e^t + C}}$

(B) $\int e^{8t} \, dt$ $= \underline{\underline{\frac{e^{8t}}{8} + C}}$

$k = 8$

(C) $\int 3^{-2x} \, dx$ $= \underline{\underline{\frac{3^{-2x}}{-2(\ln 3)} + C}}$

$k = -2$

$a = 3$

Indefinite Integral of x^{-1}:

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

$\frac{d}{dx} \ln|x| = \frac{1}{x}$

Example. Find the integral

$$\int -\frac{2}{x}\,dx = -2 \int \frac{1}{x}\,dx = \underline{\underline{-2\ln|x| + C}}$$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

Example. Find the cost function for the following marginal cost function, $C'(x) = 6x - 7$ with fixed cost \$5.

$\int f(x)\,dx = F(x) + C$ where $F'(x) = f(x)$

$$C(x) = \int (6x-7)\,dx = 6\int x\,dx - 7\int dx$$

$$= 6 \cdot \frac{x^{1+1}}{1+1} - 7 \cdot x + K$$

$$C(x) = 3x^2 - 7x + K$$

K is a constant

$C(0) = 5$
fixed cost

$5 = 3 \cdot 0^2 - 7 \cdot 0 + K$

$5 = 0 + K$

$K = 5$

$$\underline{\underline{C(x) = 3x^2 - 7x + 5}}$$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Example. Given the marginal revenue function, $R'(q) = 500 - 4e^{0.0005q}$.

$\int e^{kx} dx = \frac{e^{kx}}{k} + C$

(A) Find the revenue function, $R(q)$

$$R(q) = \int 500 - 4e^{0.0005q}\, dq$$

$$= 500\int dq - 4\int e^{0.0005q}\, dq = 500q - 4 \cdot \frac{e^{0.0005q}}{0.0005} + C$$

$$R(q) = 500q - 8000e^{0.0005q} + C$$

$R(0)=0$ (no items sold, means no revenue)

$$0 = 500 \cdot 0 - 8000 \cdot e^{0.0005 \cdot 0} + C$$

$$0 = -8000 + C$$

$$\Rightarrow C = 8000$$

$$R(q) = \underline{500q - 8000e^{0.0005q} + 8000}$$

(B) Find the demand function, $p(q)$.

$$R = p \cdot q$$

$$500q - 8000e^{0.0005q} + 8000 = p \cdot q$$

Divide both sides by q.

$$p = \frac{500q - 8000e^{0.0005q} + 8000}{q}$$