6.3: Further business applications: Economic lot size; economic order quantity; elasticity of demand.

We will consider the following business applications:

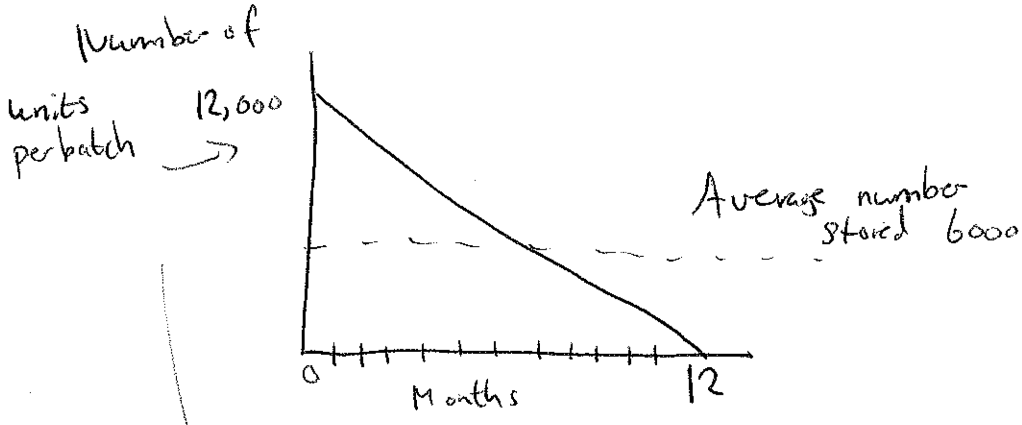
1. Economic lot size. (A manufacturer must determine the production lot size that will result in minimum production and storage cost.)

2. Economic order quantity. (A purchaser must decide what quantity of an item to order that will result in minimum reordering and storage cost.)

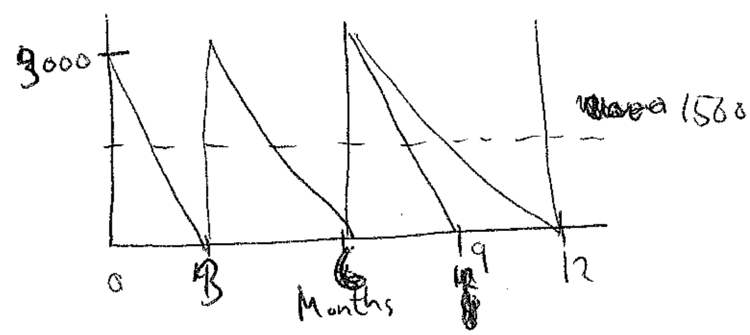
3. Elasticity of demand. (How changes in price per item of a product affects the quantity of the product demanded.)

Economic lot size

Suppose that a company manufactures a constant number of units of a product per year and that the product can be manufactured in several batches of equal size throughout the year. The company has to consider the setup costs and warehouse costs. Many small batches increase set up cost. Large batches minimize setup costs but results in higher warehouse costs. We will use calculus to find the number of units that should be manufactured in each batch in order to minimize the total cost. This number is called the **economic lot size**.



All 12000 units are made in one batch per year.



Manufactures 3000 units in each batch So 4 batches will be made at equal time intervals during the year.

Variables

$q =$ number of units in each batch.
$k =$ cost of storing one unit of the product for one year.
$f =$ fixed setup cost to manufacture the product.
$g =$ cost of manufacturing a single unit of the product.
$M =$ total number of units produced annually.

The company has two types of cost:
1. Cost to manufacture the product.
2. Cost to store the finished product.

q units are produced in each batch and each batch has a fixed cost f and cost g per unit.

1.
The manufacturing cost per batch is: $f + g \cdot q$

The number of batches per year is: $\frac{M}{q}$

The total annual manufacturing cost is : $(f + g \cdot q) \cdot \frac{M}{q} = \frac{f \cdot M}{q} + g \cdot M$

2.
Since demand is constant, the inventory goes down linearly from q to 0 with an average of inventory of $\frac{q}{2}$ units per year. The cost of storing one unit of the product for a year is k.

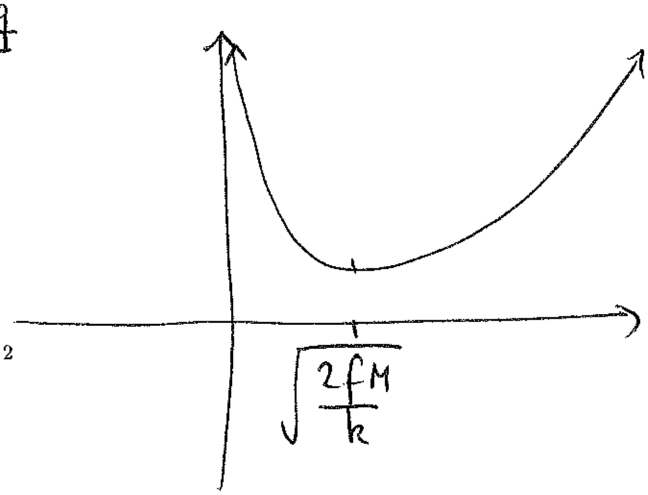
The total storage cost is: $k \cdot \frac{q}{2} = \frac{k \cdot q}{2}$

The total production cost = manufacturing cost + storage cost.

The **total production cost** of producing M units in batches of size q is,

$T(q) = \frac{f \cdot M}{q} + g \cdot M + \frac{k \cdot q}{2}$

The domain of T is $(0, \infty)$



Find the value of q that will minimize $T(q)$.

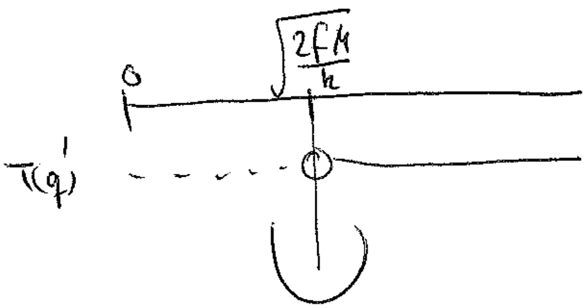
f, g, k, M are constants

$$T'(q) = -\frac{f \cdot M}{q^2} + \frac{k}{2} = 0$$

$$\frac{k}{2} = \frac{f \cdot M}{q^2}$$

$$\frac{k}{2} q^2 = f \cdot M$$

$$q^2 = \frac{2 \cdot f \cdot M}{k} \qquad q = \sqrt{\frac{2 \cdot f \cdot M}{2}} \text{ one critical point}$$



$q = \sqrt{\frac{2 \cdot f \cdot M}{2}}$ is a relative min.

By the critical point thm, there is an absolute min at $q = \sqrt{\frac{2 \cdot f \cdot M}{2}}$.

Economic lot size

$$q = \sqrt{\frac{2fM}{k}}$$

is the economic lot size that minimizes total production cost,

$$T(q) = \frac{fM}{q} + gM + \frac{kq}{2}.$$

The number of batches produced per year is $\frac{M}{q}$.

Example. A manufacturer has a steady annual demand for 16,000 cases of sugar. It costs \$8 to store one case for one year, \$32 in setup cost to produce each batch, and \$14 to produce each case.

(A) Find the number of cases per batch that should be produced to minimize total production cost.

$k = 8 \qquad M = 16{,}000$

$f = 32 \qquad g = 14$

$$q = \sqrt{\frac{2 \cdot f \cdot M}{k}} = \sqrt{\frac{2 \cdot 32 \cdot 16000}{8}} \approx 357.8$$

$$T(q) = \frac{f \cdot M}{q} + g \cdot M + \frac{k \cdot q}{2} = \frac{32 \cdot 16{,}000}{q} + 14 \cdot 16000 + \frac{8 \cdot q}{2}$$

$$= \frac{512000}{q} + 224000 + 4q$$

$$T(357) = \frac{512000}{357} + 224000 + 4 \cdot 357$$

$$= 226862.1737$$

$$T(358) = 226862.1676$$

$T(358) < T(357)$

$q = 358$ will give a minimum total production cost

Cases per batch is $\underline{\underline{358}}$

(B) What is the minimum cost?

Min cost is

$T(358) = 226862.1676$

$\underline{\underline{\text{dollar}}}$

(C) Find the number of batches of sugar that should be manufactured annually.

$$\frac{M}{q} = \frac{16000}{358} = 44.69 \approx 45$$

$\underline{\underline{\text{45 batches}}}$ annually

Economic order quantity:

Now we will consider the situation when the company reorder an item that is used at a constant rate throughout the year.

The company using the product must decide:
1. how often to order,
2. how many units to request each time an order is placed.

In order words, the company must identify the **economic order quantity**.

Variables

$q =$ number of units to order each time.
$k =$ cost of storing one unit of the product for one year.
$f =$ fixed cost to place an order.
$M =$ total number of units needed each year.

Total cost=Storage cost+Reorder cost

Reorder cost: f

The number of orders placed annually is: $\frac{M}{q}$

The **total cost** of ordering M units per year with q units each time is,

Reorder cost: $f \cdot \frac{M}{q}$

$$T(q) = f \cdot \frac{M}{q} + k \cdot \frac{q}{2}$$

Storage cost: $k \cdot \frac{q}{2}$

Store on average inventory of $\frac{q}{2}$

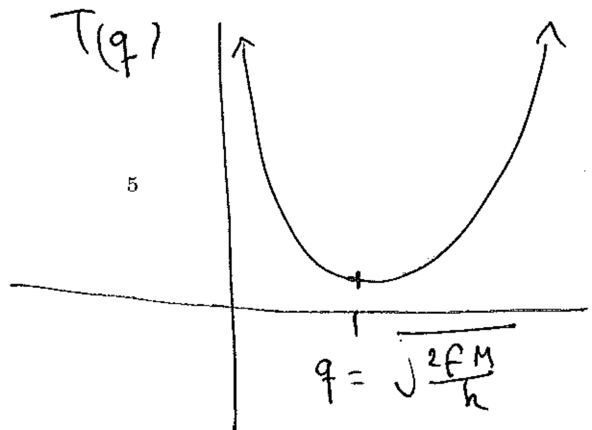
Economic order quantity

$$q = \sqrt{\frac{2fM}{k}}$$

is the economic order quantity that minimizes total ordering cost,

$$T(q) = \frac{fM}{q} + \frac{kq}{2}.$$

The number of orders placed annually is $\frac{M}{q}$.



$$T(q) = \frac{f \cdot M}{q} + \frac{k \cdot q}{2} \qquad \text{min at } q = \sqrt{\frac{2 \cdot f \cdot M}{k}}$$

Example. A large camera store sells 20,000 batteries annually. It costs 15 cents to store one battery for one year, and $12 to place a reorder.

(A) Find the number of batteries that should be ordered each time.

$M = 20,000$

$f = 12$

$k = 0.15$

$$q = \sqrt{\frac{2 \cdot f \cdot M}{k}} = \sqrt{\frac{2 \cdot 12 \cdot 20,000}{0.15}} \approx 1788.9$$

$T(1788) = 268.32819$

$T(1789) = 268.32816$

$$T(q) = \frac{12 \cdot 20,000}{q} + \frac{0.15 \cdot q}{2}$$

$$T(q) = \frac{240,000}{q} + 0.075q$$

(B) How many times per year should the batteries be ordered?

$T(1789) < T(1788)$

So $q = 1789$ gives a minimum total cost.

Hence we should reorder $\underline{1789}$ batteries each time

B)

$$\frac{M}{q} = \frac{20,000}{1789} \approx 11.2 \qquad \frac{12}{11.2} \approx 1.07$$

about once every $\underline{1.07}$ months

Elasticity of Demand

Recall that the change in price per unit of a product affects the demand.

We will now analyze the measure of sensitivity of demand to changes in price.

The ratio of percent change in demand to percent change in price is:

$$\frac{\Delta q/q}{\Delta p/p},$$

where,
q=quantity demanded,
p = price per item,
Δq = change in q,
Δp = change in p.

This ratio is negative. Why? $p, q > 0$ Δq and Δp have opposite sign

An increase in price causes a decrease in demand

We will rewrite the ratio:

If the absolute value of this ratio is ~~greater than 1~~ large, then a relatively small increase in price causes a relatively large decrease in demand.

Elasticity of Demand

Let $q = f(p)$, where q is demand at a price p. The **elasticity of demand** is

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

.

- Demand is inelastic if $E < 1$.
- Demand is elastic if $E > 1$.
- Demand has unit elasticity if $E = 1$.

An **inelastic** demand is one in which a change in price leads to a **small** change in quantity demanded.
Examples of products with an inelastic demand are necessities like food and fuel.

An **elastic** demand is one in which a change in price leads to a **large** change in quantity demanded.
Examples of products with an elastic demand are products that are purchased infrequently which can be postponed if price rises.

Unit elasticity means that the percentage changes in price and demand are relatively equal.

$$E = -\frac{p}{q} \quad \frac{dq}{dp}$$

Example. Find the elasticity of demand E for the demand function

$$q = 200 - 0.5p^2$$

for $p = \$7$ per item.
Is the demand elastic, inelastic, or neither?

$$\frac{dq}{dp} = -0.5 \cdot 2p = -p$$

$$E = -\frac{p}{q} \cdot \frac{dq}{dp}$$

$$= \frac{-p}{200-0.5p^2} \cdot (-p) = \frac{p^2}{200-0.5p^2}$$

$$E(7) = \frac{7^2}{200 - 0.5 \cdot 7^2} = \underline{0.28} < 1$$

The elasticity of demand is <u>inelastic</u>

Elasticity is related to total revenue $R = pq$.

$$\frac{dR}{dp} = q\,(1-E)$$

$$\frac{dR}{dp} = q(1-E)$$

$E < 1$

$\frac{dR}{dp} > 0$

$E > 1$

$\frac{dR}{dp} < 0$

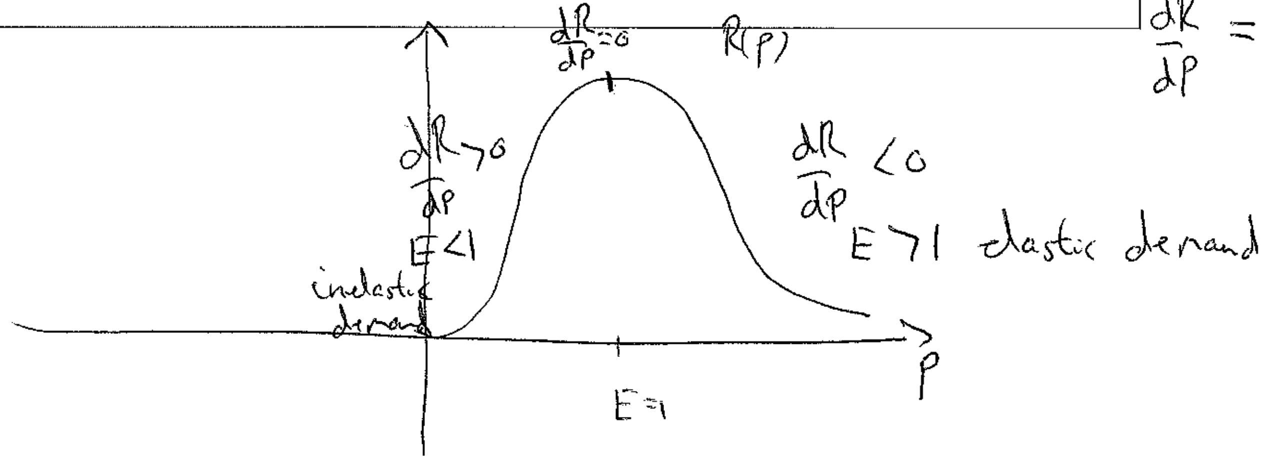
$E = 1$

$\frac{dR}{dp} = 0$

Revenue and Elasticity

Total revenue R is increasing if $\frac{dR}{dp} > 0$ which corresponds to $E < 1$
Total revenue R is decreasing if $\frac{dR}{dp} < 0$ which corresponds to $E > 1$
Total revenue R is optimized if $\frac{dR}{dp} = 0$ which corresponds to $E = 1$

- If the demand is inelastic, total revenue increases as price increases.
- If the demand is elastic, total revenue decreases as price increases.
- Total revenue is maximized at the price where demand has unit elasticity.



Example. Given the demand function

$$q = 200 - 0.5p^2.$$

Find the values of p (if any) at which total revenue is maximized.

From last problem

$$E = \frac{p^2}{200 - 0.5p^2} = 1$$

$$p^2 = 200 - 0.5p^2$$

$$1.5p^2 = 200$$

$$p^2 = \frac{200}{1.5}$$

$$p = \sqrt{\frac{200}{1.5}} \approx 11.5$$

When the price is 11.5 dollar, the revenue is a max