6.2: Applications of Extrema

In this chapter we will apply calculus to maximum and minimum problems.

Solving an Applied Extrema Problem

1. If possible sketch a picture including the variables.

2. Decide on the variable that must be maximized or minimized. Express that variable as a function of **one** other variable.

3. Find the domain of the function.

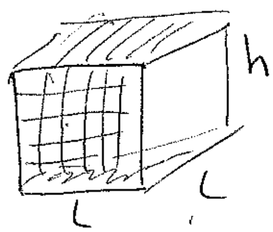
4. Find the critical points for the function from Step 2.

5. If the domain is a closed interval, evaluate the function at the endpoints and at each critical number to see which yields the absolute maximum or minimum. If the domain is an open interval, apply the critical point theorem when there is only one critical number. If there is more than one critical number, evaluate the function at the critical numbers and find the limit as the endpoints of the interval are approached to determine if an a absolute maximum or minimum exists at one of the critical points.

Example. A closed box with a square base is to have a volume of 10,000 cm^3. The material for the top and bottom of the box costs \$2 per square centimeter, while the material for the sides costs \$3 per square centimeter. Find the dimensions of the box that will lead to the minimum total cost. What is the minimum total cost?

$V = 10{,}000$

top and bottom
\$2 per cm^2



$V = L \cdot L \cdot h = 10{,}000$

$L^2 \cdot h = 10{,}000$

$h = \frac{10{,}000}{L^2}$

sides
\$3 per cm^2

Total cost:

Area of top and bottom: $2 \cdot L \cdot L = 2L^2$ cm^2

Area of sides: $4 \cdot L \cdot h$ cm^2

Material for the top and bottom costs: \$ $2 \cdot 2L^2 = 4L^2$ dollar

Material for the sides: \$ $3 \cdot 4L h = 12 L \cdot h$ dollar

Total cost: C = Cost of top and bottom + Cost of sides

$C = 4L^2 + 12L \cdot h$

$C(L) = 4L^2 + 12L \cdot \left(\frac{10{,}000}{L^2}\right)$ Cost as a function of L.

$C(L) = 4L^2 + \frac{120{,}000}{L}$

Want to find the absolute min of $C(L)$.

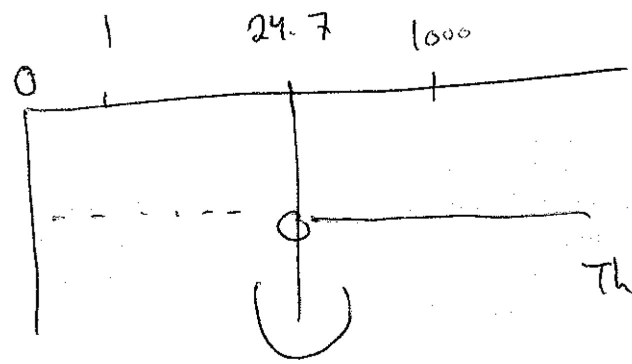
$$C'(l) = 8l - \frac{120{,}000}{l^2} = \frac{8l^3 - 120000}{l^2}$$

$$= \frac{8(l^3 - 15000)}{l^3} = 0$$

$$l^3 - 15000 = 0$$

$$l = (15000)^{\frac{1}{3}} \approx \underline{24.7}$$

$l > 0$



$C'(l) = \frac{8(l^3 - 15000)}{l^3}$

$C'(1) = \frac{8(1^3 - 15000)}{1^3} < 0$

$C'(1000) = \frac{8((1000)^3 - 15000)}{(1000)^3} > 0$

There is a relative min at $l = 24.7$.

By the critical number theorem, $\underline{l = 24.7}$ gives an absolute min.

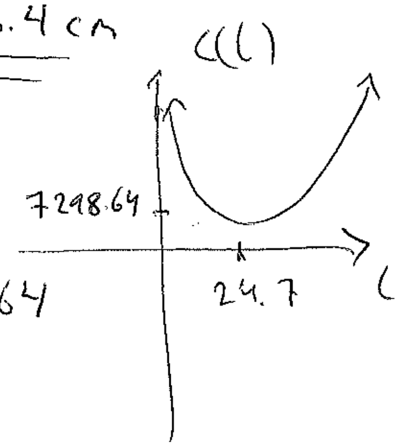
$$h = \frac{10{,}000}{l^2} = \frac{10000}{(24.7)^2} \approx \underline{16.4}$$

Dimensions that minimize the cost is $l = \underline{24.7}$ cm and $h = \underline{\underline{16.4}}$ cm

$$C(l) = 4l^2 + \frac{120000}{l}$$

$$C(24.7) = 4 \cdot (24.7)^2 + \frac{120000}{24.7} \approx 7298.64$$

Minimum total cost is $\$\ \underline{\underline{7298.64}}$



Example. In planning a cafe, it is estimated that a profit of $4 per seat will be made if the number of seats is no more than 30 inclusive. On the other hand, the profit on each seat will decrease by 10 cent for each seat above 30.
(A) Find the number of seats that will produce the maximum profit.
(B) What is the maximum profit?

A) ~~X = the number of additional tables~~

X = the number of Seats

Profit is 4 dollars per seat for $0 < X \leq 30$

Profit (in dollars) is $4 - 0.1(X-30)$ per Seat for $X > 30$.

Total profit for X seats is

$P(x) = [4 - 0.1(x-30)] \cdot x$

$= (4 - 0.1x + 3)x$

$= 7x - 0.1x^2$ $\qquad P'(x) = 7 - 0.2x$

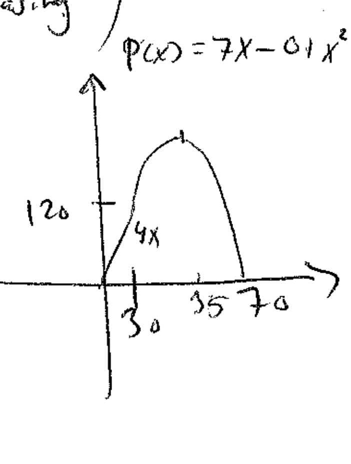
$X = 35$

We expect that the total number of seats which makes the total profit a maximum will be greater than 30 because after 30 the profit is still increasing

$4 \cdot 30 = 120$

$P(35) = 7 \cdot 35 - 0.1 \cdot (35)^2 = 122.5$

$\lim_{x \to \infty} P(x) = -\infty$



B) $P(35) = 122.5$ \$

Max profit is \$122.5