## 6.1: Absolute extrema

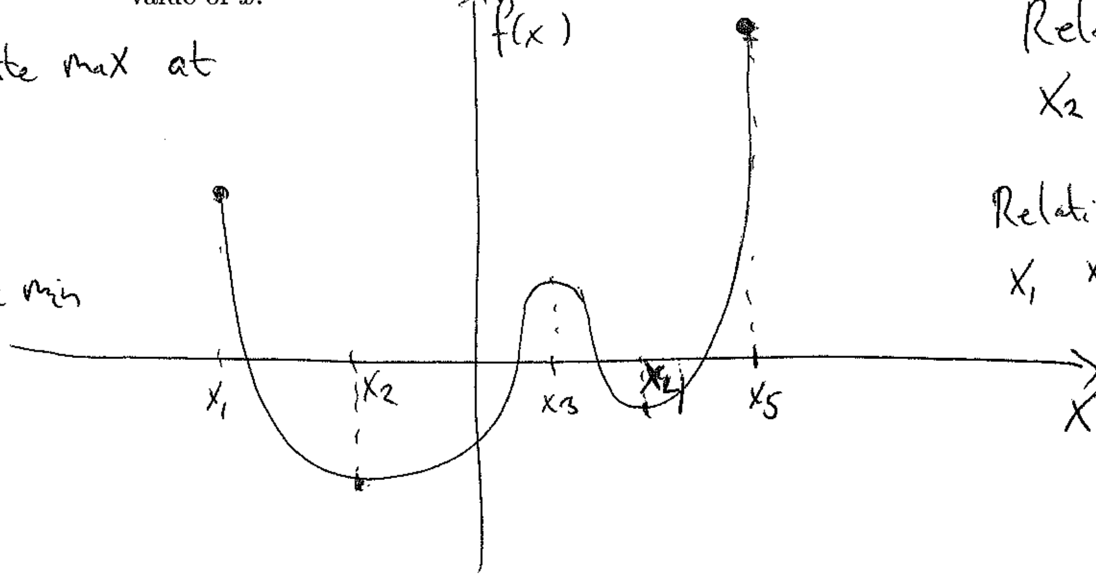
- The largest possible value of a function is called the **absolute maximum**.
- The smallest possible value of a function is called the **absolute minimum**.
- A function can have several relative maxima or relative minima, but it never has more than one **absolute maximum** or **absolute minimum**.
- The **absolute maximum** or **absolute minimum** might occur at more than one value of $x$.

Absolute max at $X_5$

Relative min at $X_2$, $X_4$

Relative max at $X_1$ $X_3$, $X_5$

Absoulute min at $X_2$




**Absolute maximum or minimum:**

Let $f$ be a function defined on some interval. Let $c$ be a number in the interval. Then $f(c)$ is the **absolute maximum** of $f$ on the interval if

$$f(x) \leq f(c)$$

for every $x$ in the interval, and $f(c)$ is the **absolute minimum** of $f$ on the interval if

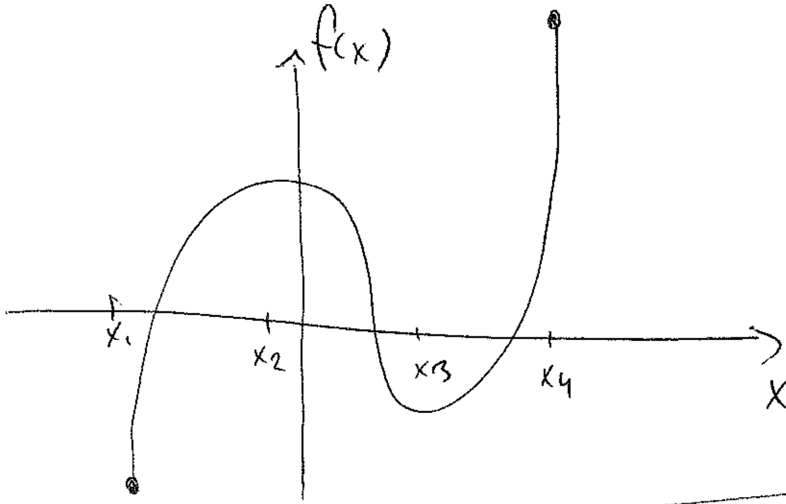
$$f(x) \geq f(c)$$

for every $x$ in the interval.

A function has an **absolute extremum** (plura:**extrema**) at $c$ if it has either as absolute maximum or an absolute minimum there.

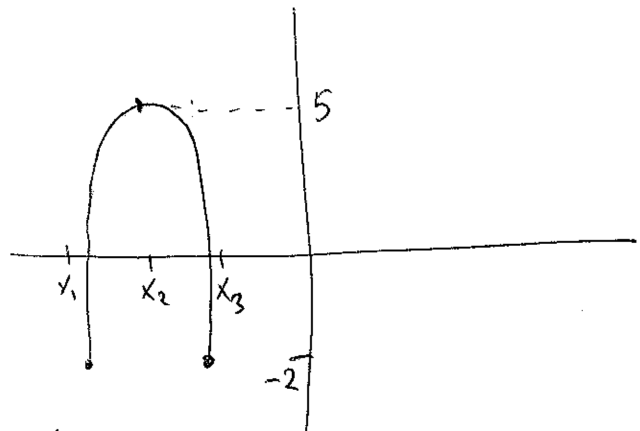
**Notice:**
Just like a relative extremum, an absolute extremum is a $y$-value.

**Example.**




Absolute max at $x = x_4$

Absolute min at $x = x_1$




Find the absolute max and min and where they are located.

Absolute max at $x = x_2$.

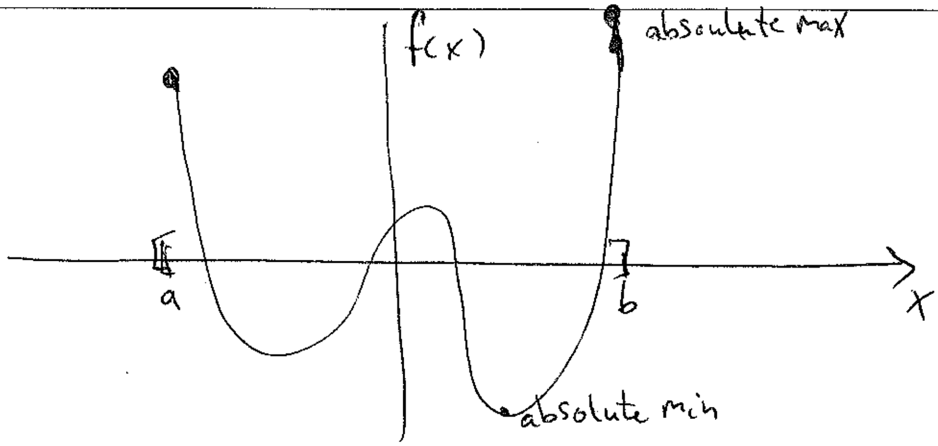
Absolute max is $\underline{\underline{5}}$

Absolute min at $x_1$ and $x_3$.

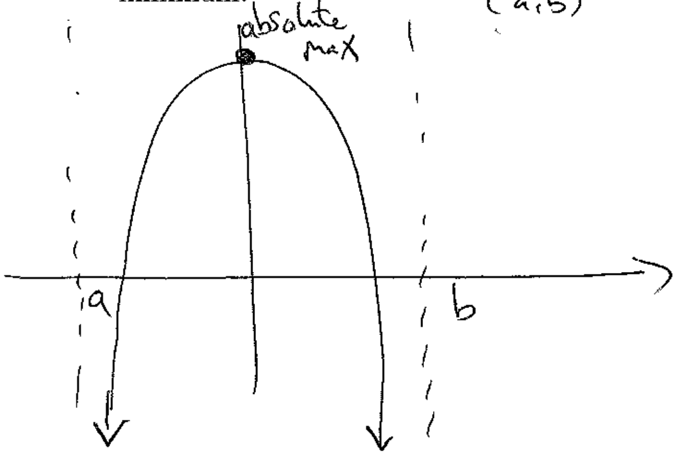
Absolute min is $-\underline{\underline{2}}$

**Extreme Value Theorem:**

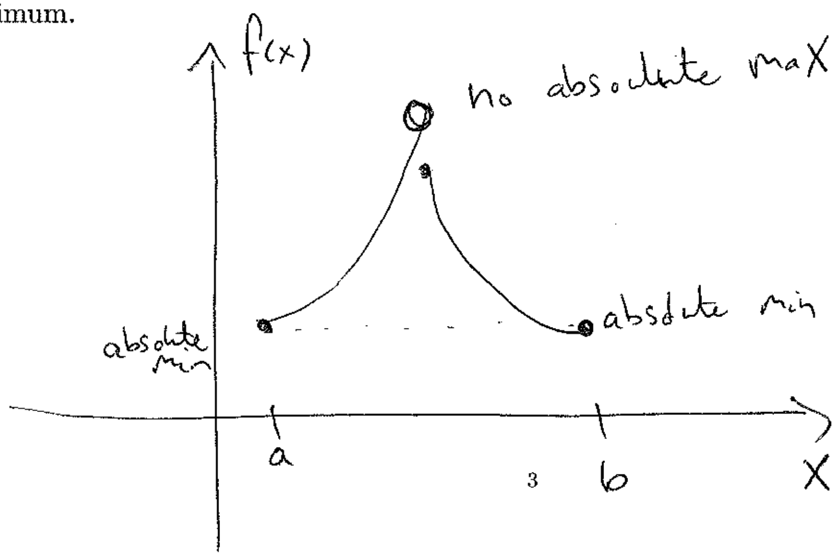
A function $f$ that is continuous on a closed interval $[a, b]$ will have both an absolute maximum and an absolute minimum on the interval.




A continuous function on an **open** interval may or may not have an absolute maximum or minimum. (a,b)




A discontinuous function on a closed interval may or may not have an absolute minimum or maximum.



**Finding Absolute Extrema:**

To find absolute extrema for a function $f$ that is continuous on a closed interval $[a, b]$:
1. Find all critical numbers for $f$ in $(a, b)$.
2. Evaluate $f$ for all critical numbers in $(a, b)$.
3. Evaluate $f$ for the endpoints $a$ and $b$ of the interval $[a, b]$.
4. The largest value found in Step 2 or 3 is the absolute maximum for $f$ on $[a, b]$, and the smallest value found is the absolute minimum for $f$ on $[a, b]$.

**Critical Point Theorem:**

Suppose a function $f$ is continuous on an interval $I$ and that $f$ has **exactly** one critical number in the interval $I$, located at $x = c$.
If $f$ has a relative maximum at $x = c$, then this relative maximum is the absolute maximum of $f$ on the interval $I$.
If $f$ has a relative minimum at $x = c$, then this relative minimum is the absolute minimum of $f$ on the interval $I$.

**Example.** Find the absolute extrema of the function if they exist, as well as all values of $t$ where they occur.

$$g(t) = 4t + \frac{16}{t^2} + 1 \quad \text{for} \quad t > 0.$$

$$16t^{-2} = \frac{16}{t^2}$$

$$g'(t) = 4 + 16(-2)\cdot t^{-3} = 4 - \frac{32}{t^3} = \frac{4t^3 - 32}{t^3} = 0$$

$$= \frac{4(t^3-8)}{t^3} = 0$$

$$4(t^3-8) = 0$$

$$t^3 - 8 = 0$$

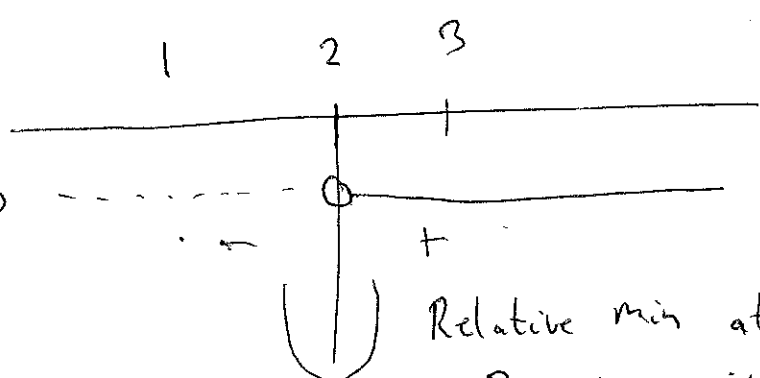
$$t^3 = 8$$

$$t = 8^{\frac{1}{3}} = 2$$

Critical number is $\underline{t=2}$

$$g(2) = 4\cdot 2 + \frac{16}{2^2} + 1 = \underline{13}$$

$$g'(t) = \frac{4(t^3-8)}{t^3}$$




Relative min at $t=2$

By the critical point thm, $t=2$ will give a absolute min.

$$\frac{4(1-8)}{1} = -28$$

$$\frac{4(3^3-8)}{3^3} > 0$$

Absolute min $g(2) = \underline{13}$

**Example.** Find the minimum value of the average cost for the given cost function on the given interval,

$$C(x) = 81x^2 + 17x + 324 \quad \text{for} \quad 1 \le x \le 10.$$

Average cost function

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{81x^2 + 17x + 324}{x}$$

$$\bar{C}(x) = 81 \cdot x + 17 + \frac{324}{x}$$

$$\bar{C}'(x) = 81 + 324(-1 \cdot x^{-2})$$

$$\frac{324}{x} = 324 \cdot x^{-1}$$

$$= 81 - \frac{324}{x^2}$$

$$= \frac{81x^2 - 324}{x^2} = \frac{81(x^2-4)}{x^2} = \frac{81(x+2)(x-2)}{x^2} = 0$$

$x+2=0$ $\quad x = \pm 2$

$x-2=0$

Critical numbers: $x = 2$

| $x$ | $\bar{C}(x)$ |
|---|---|
| 2 | 341 |
| 1 | 422 |
| 10 | 859.4 |

$$\bar{C}(2) = \frac{81 \cdot 2^2 + 17 \cdot 2 + 324}{2} = 341$$

end points

$$\bar{C}(1) = \frac{81 \cdot 1^2 + 17 \cdot 1 + 324}{1} = 422$$

$$\bar{C}(10) = \frac{81 \cdot 10^2 + 17 \cdot 10 + 324}{10} = 859.4$$

The minimum value is $\bar{C}(2) = \underline{\underline{341}}$ dollar per item

$\bar{C}(x)$