

5.2

5.2: Relative Extrema

In this section, we will determine the maximum and minimum points on the graph.

Relative Maximum or Minimum

Let c be a number in the domain of a function f . Then $f(c)$ is a **relative (or local) maximum** for f if there exists an open interval (a, b) containing c such that

$$f(x) \leq f(c)$$

for all x in (a, b) .

Likewise, $f(c)$ is a **relative (or local) minimum** for f if there exists an open interval (a, b) containing c such that

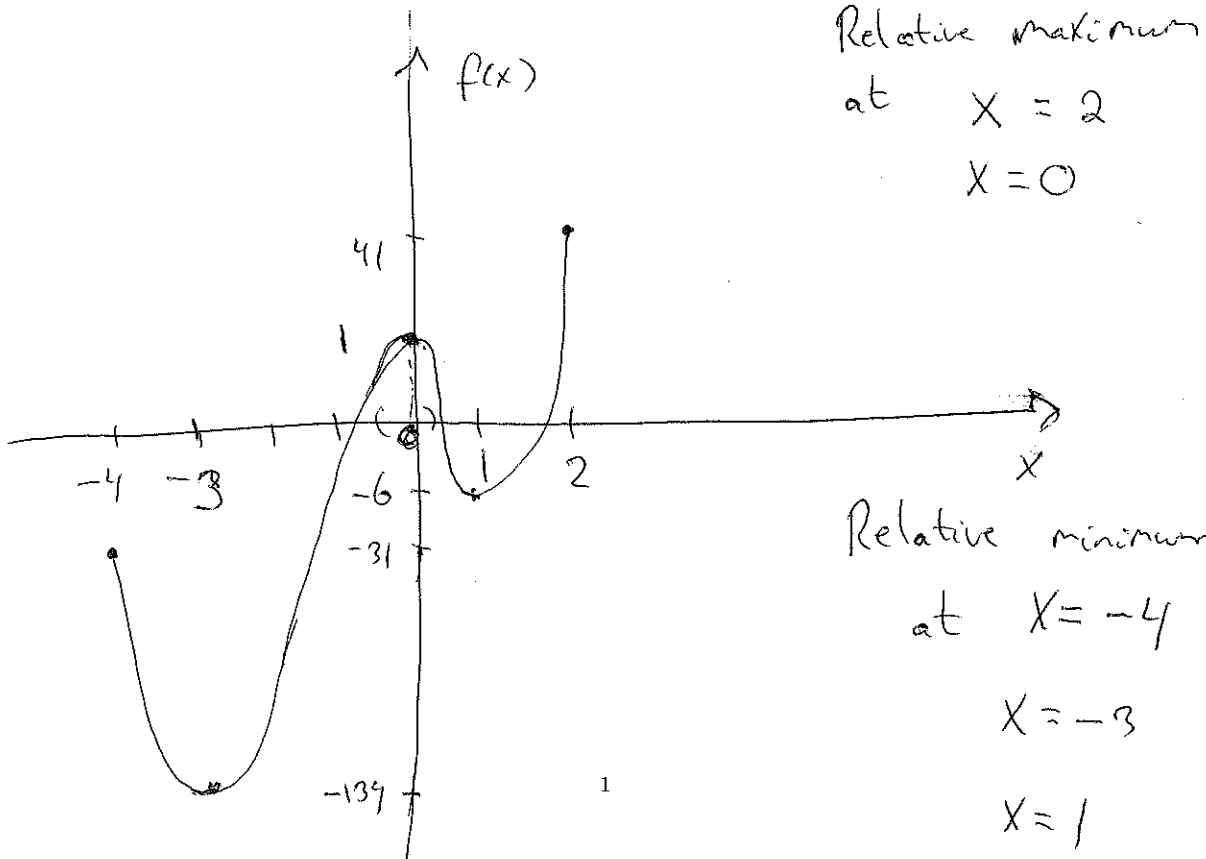
$$f(x) \geq f(c)$$

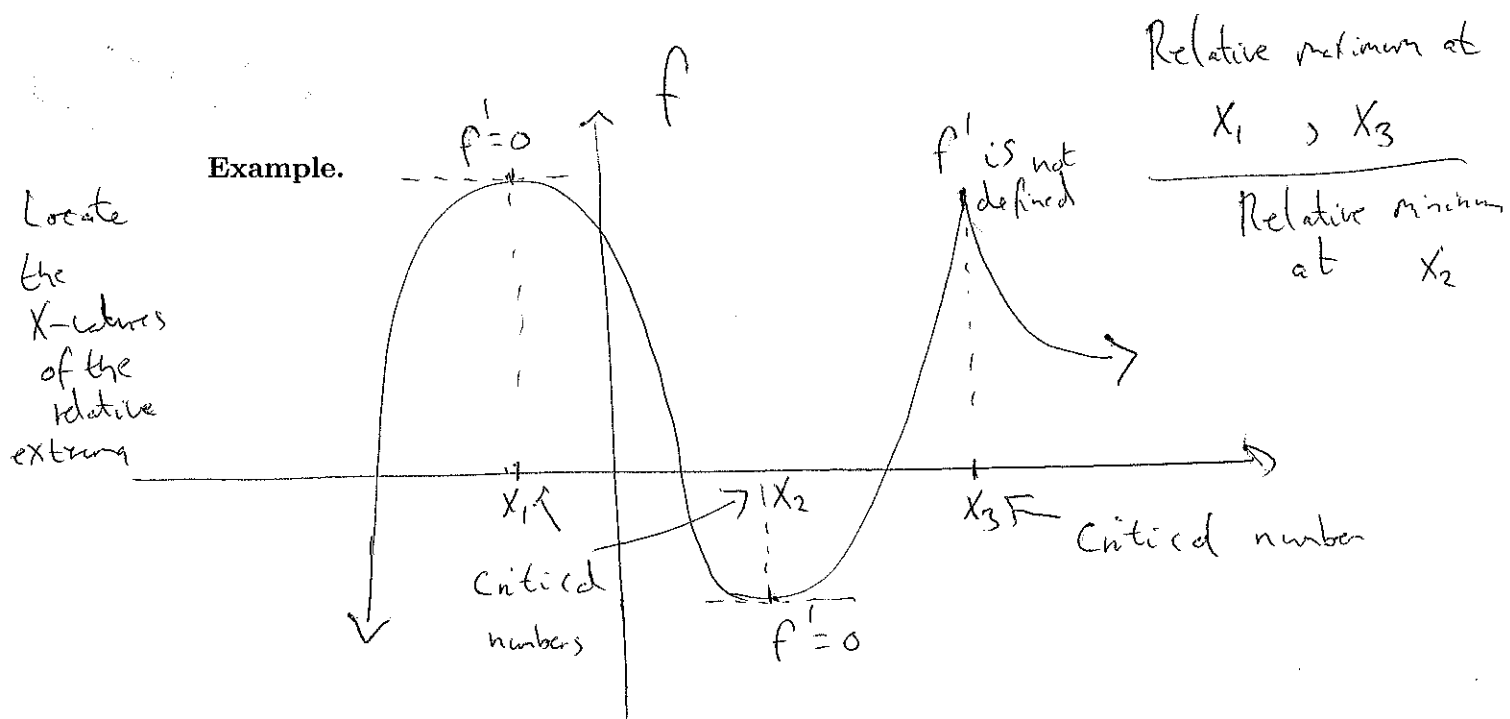
for all x in (a, b) .

A function has a **relative (or local) extremum** at c if it has either a relative maximum or a relative minimum there.

If c is an endpoint of the domain of f , we only consider x in the half-open interval that is in the domain.

Example. Draw the graph of the function $f(x) = 3x^4 + 8x^3 - 18x^2 + 1$ for $-4 \leq x \leq 2$. Identify the x -values of all points where the graph has a relative extrema:

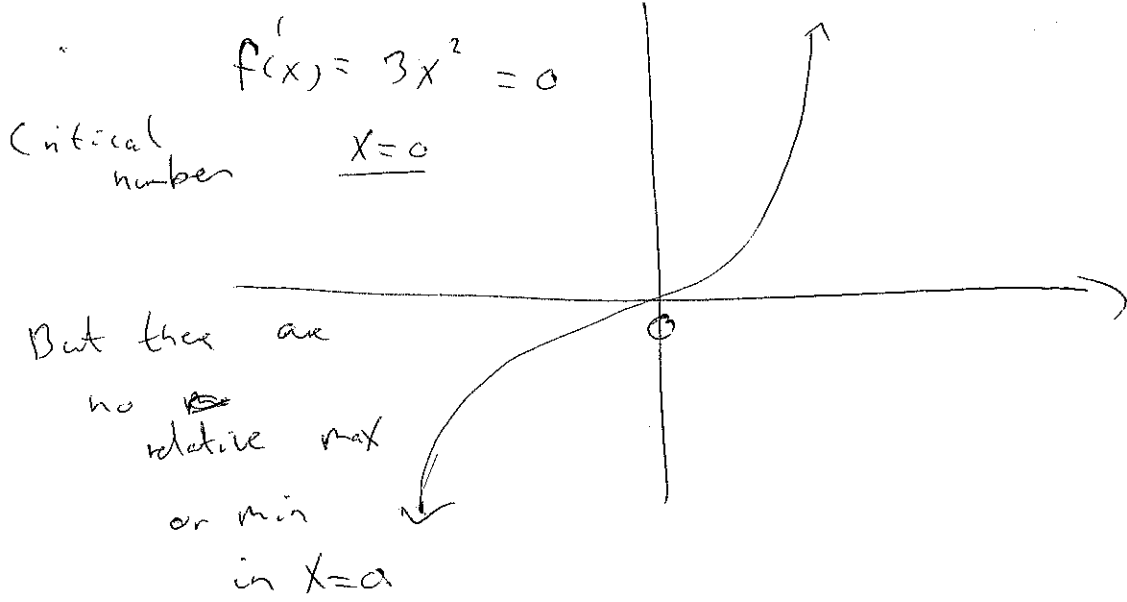




Recall that the derivative of a function is the slope of the tangent line to the graph of the function. To find relative extrema we first identify all the critical numbers and endpoints.

If a function f has a relative extremum at c , then c is a critical number or c is an endpoint of the domain.

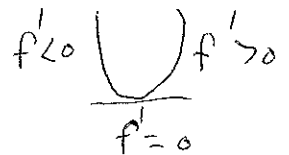
Note There is not a relative extremum at all critical numbers. An example of that is $f(x) = x^3$.



Max

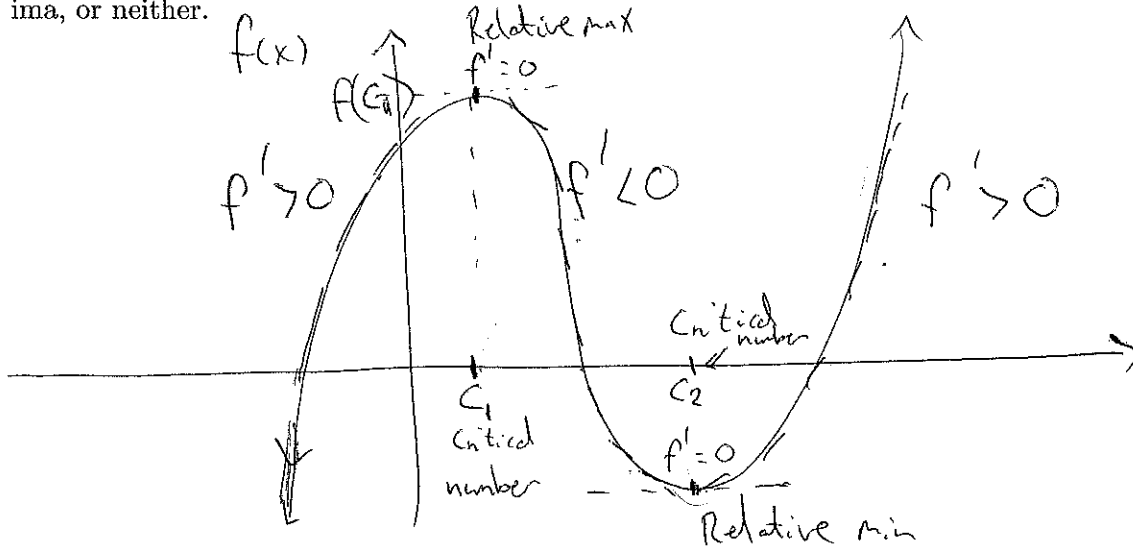


Min



First derivative test

We will now determine whether the critical numbers produce relative maxima, relative minima, or neither.

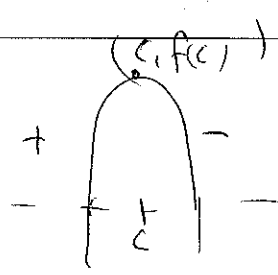
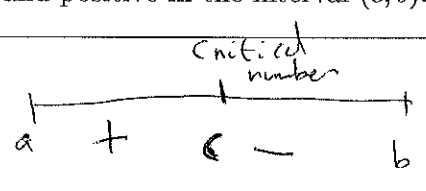


First derivative test:

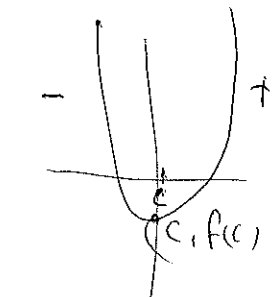
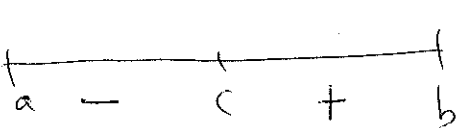
Let c be a critical number for a function f . Suppose that f is continuous on (a, b) and differentiable on (a, b) except possibly at c , and that c is the only critical number for f in (a, b) .

- $f(c)$ is a **relative maximum** of f if the derivative $f'(x)$ is positive in the interval (a, c) and negative in the interval (c, b) .
- $f(c)$ is a **relative minimum** of f if the derivative $f'(x)$ is negative in the interval (a, c) and positive in the interval (c, b) .

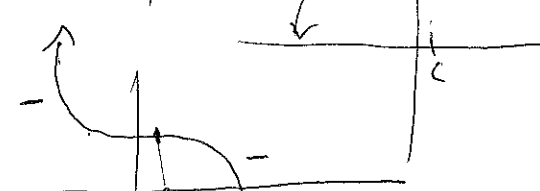
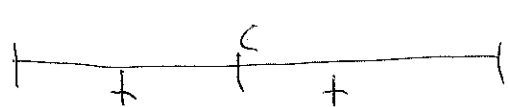
Relative max:
sign of f'



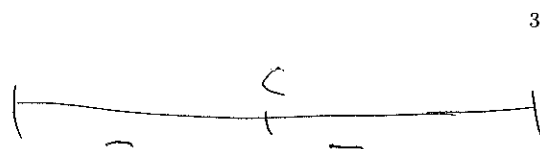
Relative min:
sign of f'



No relative extrema



No relative extrema



Example. Find all relative extrema for the following functions, as well as where each function is increasing and decreasing.

1. $f(x) = x^3 - 12x + 2$

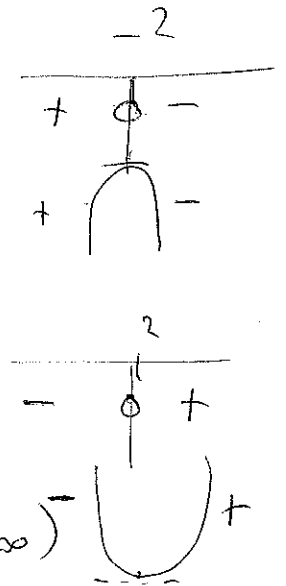
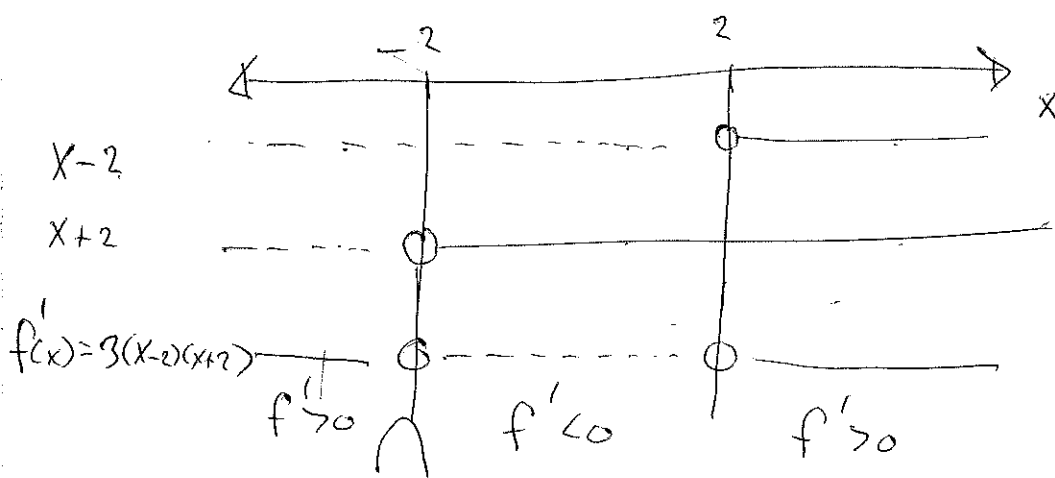
$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x-2)(x+2) = 0$$

Critical numbers: $x = \pm 2$

$$x - 2 = 0$$

$$x = \pm 2$$

$$x + 2 = 0$$



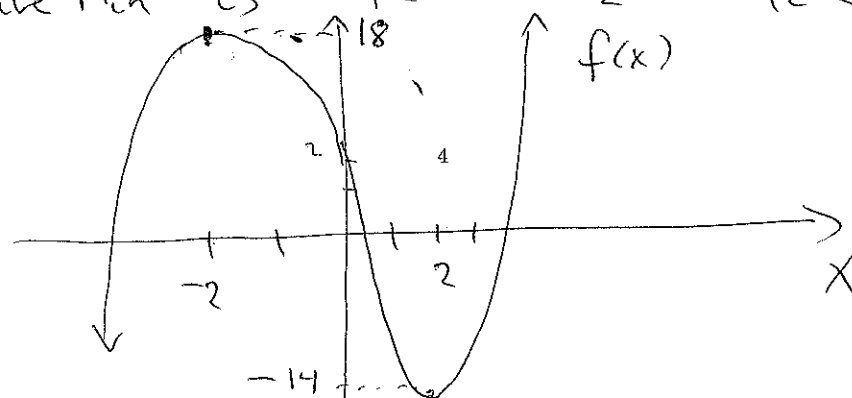
f is increasing on $(-\infty, -2)$ and $(2, \infty)$
 f is decreasing on $(-2, 2)$

f has a relative max at $x = -2$

f has a relative min at $x = 2$

The relative max is $f(-2) = (-2)^3 - 12(-2) + 2 = \underline{\underline{18}}$

The relative min is $f(2) = 2^3 - 12 \cdot 2 + 2 = \underline{\underline{-14}}$



Find all the relative extrema

2. $f(x) = x e^{4-x^2}$

$u(x) = x \quad v(x) = e^{4-x^2}$

$f'(x) = e^{4-x^2} + e^{4-x^2}(-2x) \cdot x$

$= e^{4-x^2} - 2x^2 e^{4-x^2} = e^{4-x^2}(1-2x^2)$

$= e^{4-x^2}(1-\sqrt{2}x)(1+\sqrt{2}x) = 0$

$e^{4-x^2} > 0$
for all x

$1-\sqrt{2}x = 0$ or $1+\sqrt{2}x = 0$

$1 = \sqrt{2}x$

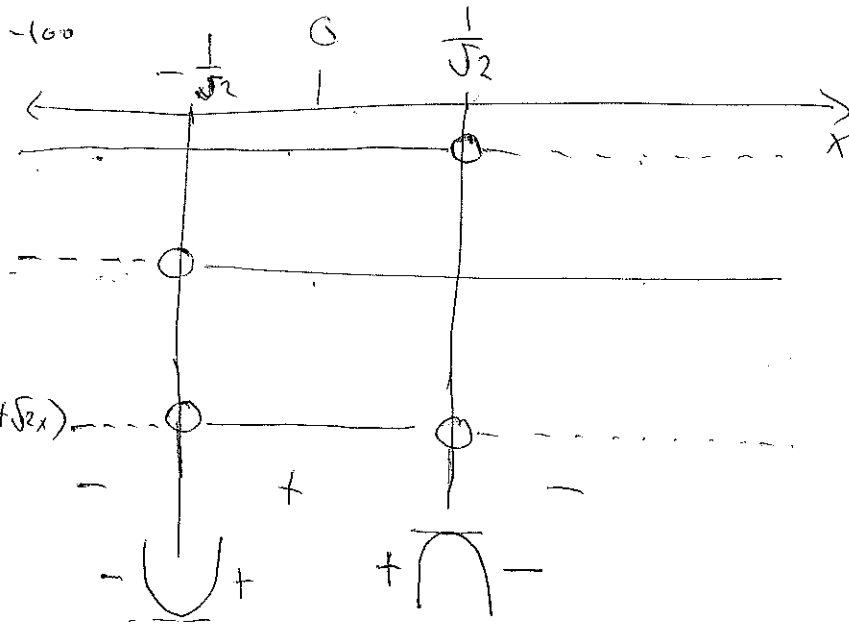
$\sqrt{2}x = -1$

$x = \frac{1}{\sqrt{2}}$

$x = -\frac{1}{\sqrt{2}}$

Critical numbers:

$\pm \frac{1}{\sqrt{2}}$



$(1-\sqrt{2}x)$
 $(1+\sqrt{2}x)$
 $f'(x) = e^{4-x^2}(1-\sqrt{2}x)(1+\sqrt{2}x)$

$1-\sqrt{2}x$
 $x=0$
 $(1-\sqrt{2} \cdot 0 = 1) > 0$
 $1+\sqrt{2}x$
 $x=-100$
 $1+\sqrt{2} \cdot (-100) < 0$

f has a relative min at $x = -\frac{1}{\sqrt{2}}$

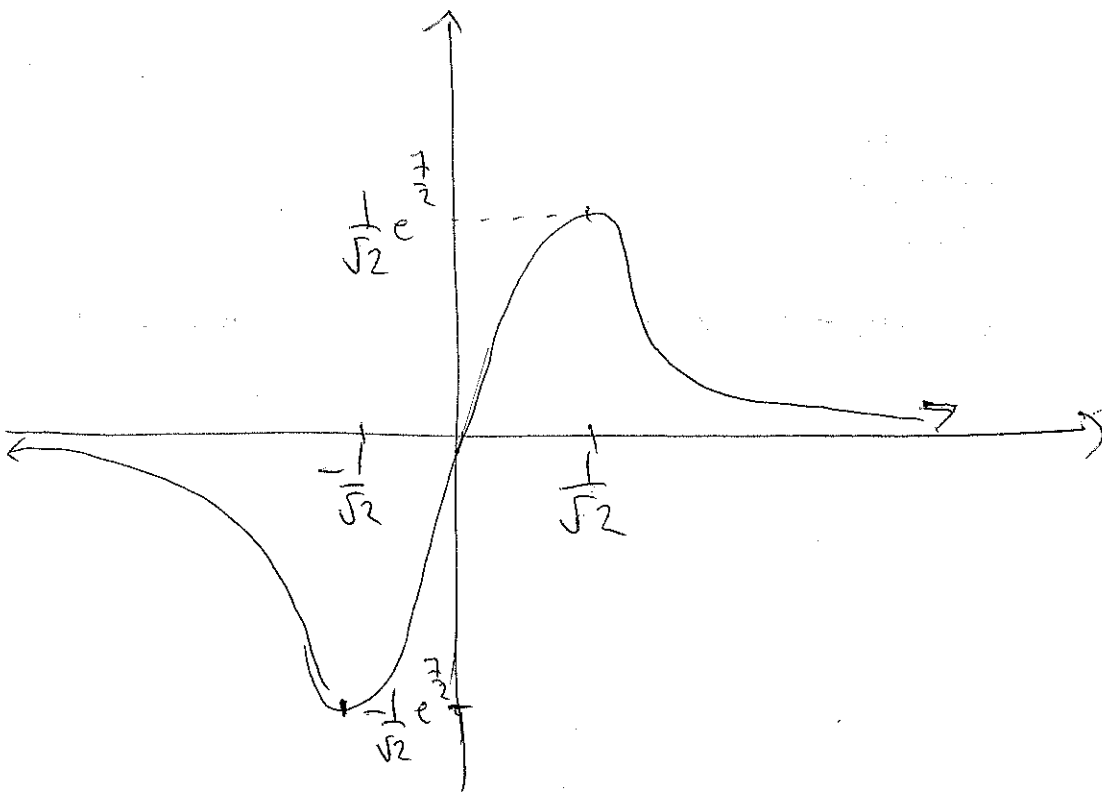
f has a relative max at $x = \frac{1}{\sqrt{2}}$

The relative min is $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} e^{4-\left(-\frac{1}{\sqrt{2}}\right)^2}$

The relative max is

$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} e^{4-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}} e^{\frac{7}{2}}$

$= -\frac{1}{\sqrt{2}} e^{4-\frac{1}{2}} = -\frac{1}{\sqrt{2}} e^{\frac{7}{2}}$



3. $f(x) = 2x - 6x^{\frac{2}{3}}$

$$f'(x) = 2 - 6 \cdot \frac{2}{3} x^{-\frac{1}{3}} = 2 - 4x^{-\frac{1}{3}} = 0$$

$$2 = 4x^{-\frac{1}{3}}$$

$$\frac{2}{4} = x^{-\frac{1}{3}}$$

$$\frac{1}{2} = x^{-\frac{1}{3}} \quad | \quad | = 2 \cdot x^{\frac{1}{3}}$$
~~$$\frac{1}{2} = x^{-\frac{1}{3}}$$~~

$$2 = x^{\frac{1}{3}}$$

$$2^3 = (x^{\frac{1}{3}})^3$$

$$x^{\frac{1}{3}} = 2$$

$$x^{\frac{1}{3}} = 2$$

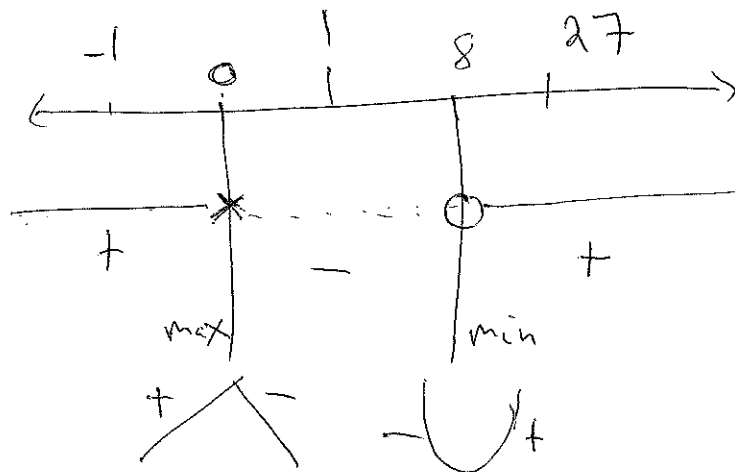
Critical number

$x=8$

$$f'(x) = 2 - \frac{4}{x^{\frac{1}{3}}} = \frac{2x^{\frac{1}{3}} - 4}{x^{\frac{1}{3}}} \quad \sqrt[3]{0} = 0$$

f' is not defined for $x=0$, but $x=0$ is in the domain of f .

Hence $x=0$ is a critical number.



$$f'(-1) = 2 - \frac{4}{(-1)^{\frac{1}{3}}}$$

$$= 2 - \frac{4}{-1} = 2 + 4 = 6 > 0$$

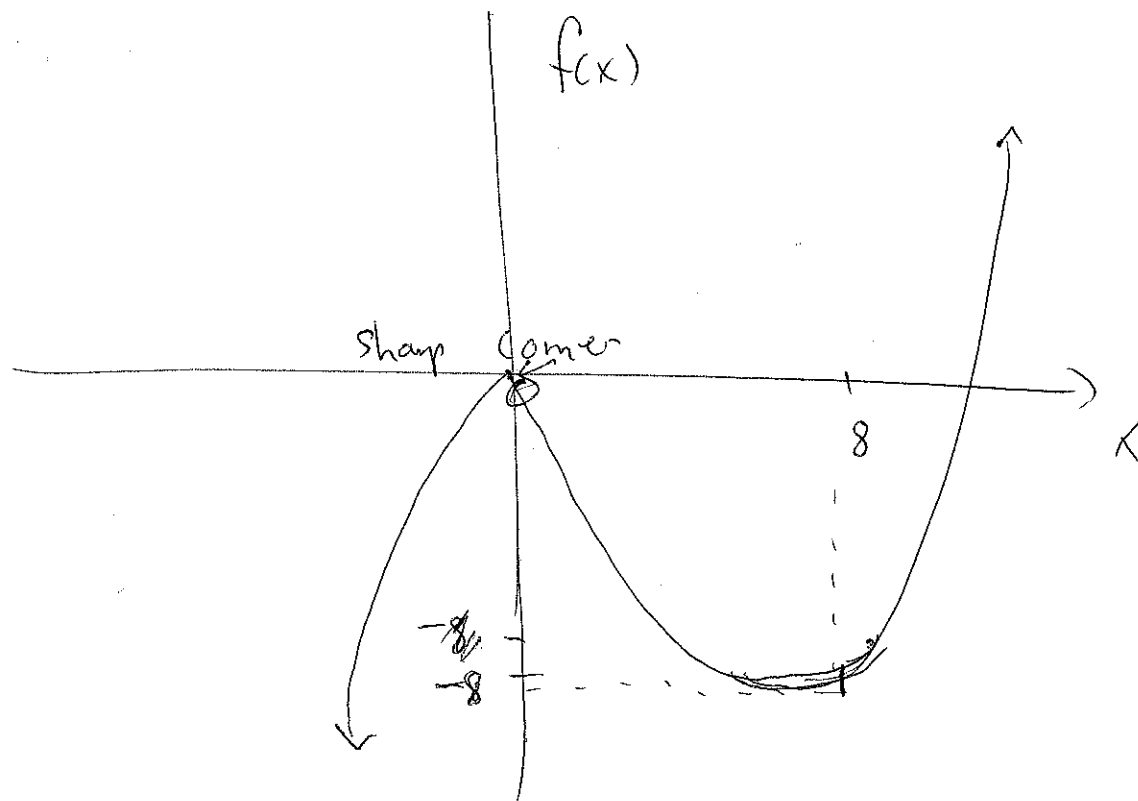
$$f'(1) = 2 - \frac{4}{1^{\frac{1}{3}}} = 2 - 4 = -2 < 0$$

$$f'(27) = 2 - \frac{4}{(27)^{\frac{1}{3}}} = 2 - \frac{4}{3} > 0$$

f has a relative max at $x=0$ and min at $x=8$

Relative max is $f(0) = 2 \cdot 0 - 6 \cdot 0^{\frac{2}{3}} = 0$

Relative min is $f(8) = 2 \cdot 8 - 6 \cdot 8^{\frac{2}{3}} = 16 - 6 \cdot 4 = -8$



Example. Let the cost to produce q units of a certain product be given by

$$C(q) = 100 + 20qe^{-0.01q}$$

and let

$$p = 40e^{-0.01q}$$

be the price per unit when q units are demanded.

(A) Find the number q of units that produces maximum profit.

$$R(q) = R(q) - C(q) \quad R(q) = p \cdot q = 40e^{-0.01q} \cdot q$$

$$P(q) = 40qe^{-0.01q} - (100 + 20qe^{-0.01q})$$

$$P(q) = 20qe^{-0.01q} - 100$$

$$u(q) = q \quad v(q) = e^{-0.01q}$$

$$P'(q) = 20 \left(1 \cdot e^{-0.01q} + e^{-0.01q} (-0.01) \cdot q \right)$$

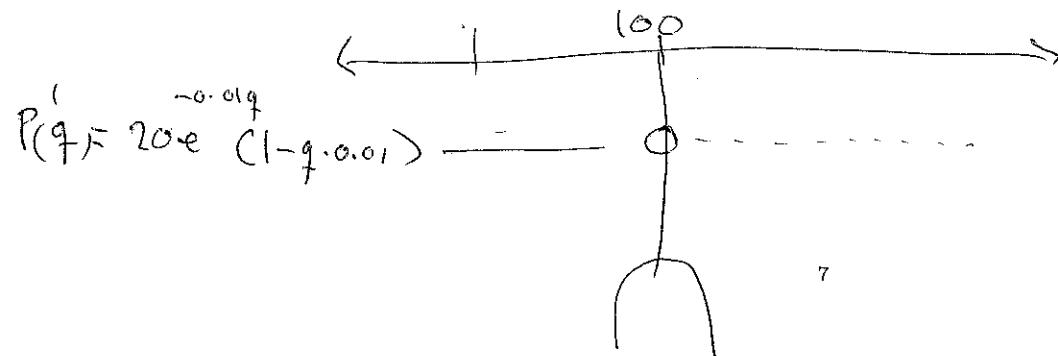
$$= 20e^{-0.01q} (1 - q \cdot 0.01) = 0$$

$$1 - q \cdot 0.01 = 0$$

$$1 = q \cdot 0.01$$

$$q = \frac{1}{0.01} = 100$$

100 units
it produces
max profit



(B) Find the price p per unit that produces maximum profit.

$$P = 40e^{-0.01 \cdot 100} = 40 \cdot e^{-1} \approx 14.7$$

When the price is \$14.7 per item,
the profit is maximum

(C) Find the maximum profit P .

$$P(100) = 20 \cdot 100 e^{-0.01 \cdot 100} - 100$$
$$\approx 635.8$$

Max profit is \$635.8