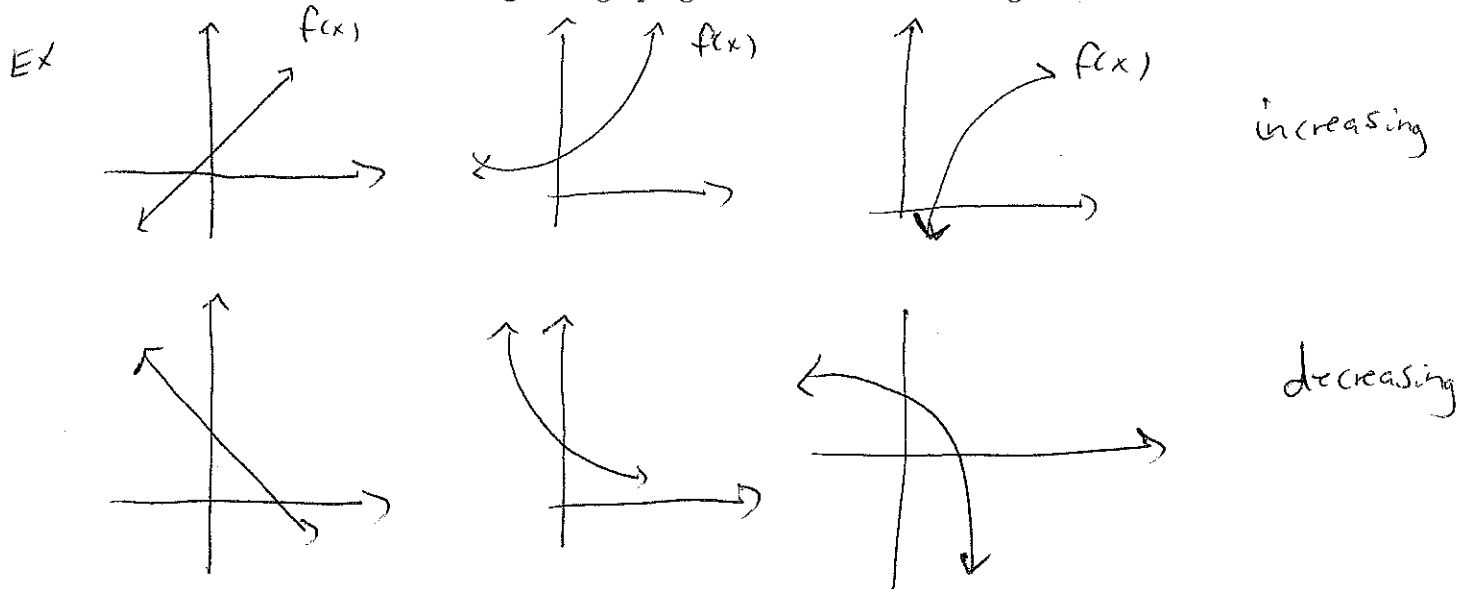


5.1: Increasing and Decreasing Functions

A function is **increasing** if its graph goes **up** from left to right.

A function is **decreasing** if its graph goes **down** from left to right.



Increasing and Decreasing Functions:

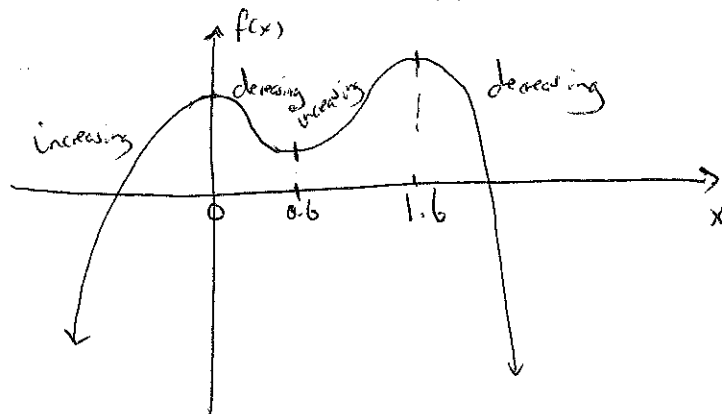
Let f be a function defined on some interval. Then for any two numbers, x_1 and x_2 in the interval, f is **increasing** on that interval if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2,$$

and f is **decreasing** on the interval if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2.$$

Example. Determine where the function $f(x) = -x^4 + 3x^3 - 2x^2 + 1$ is increasing and decreasing.



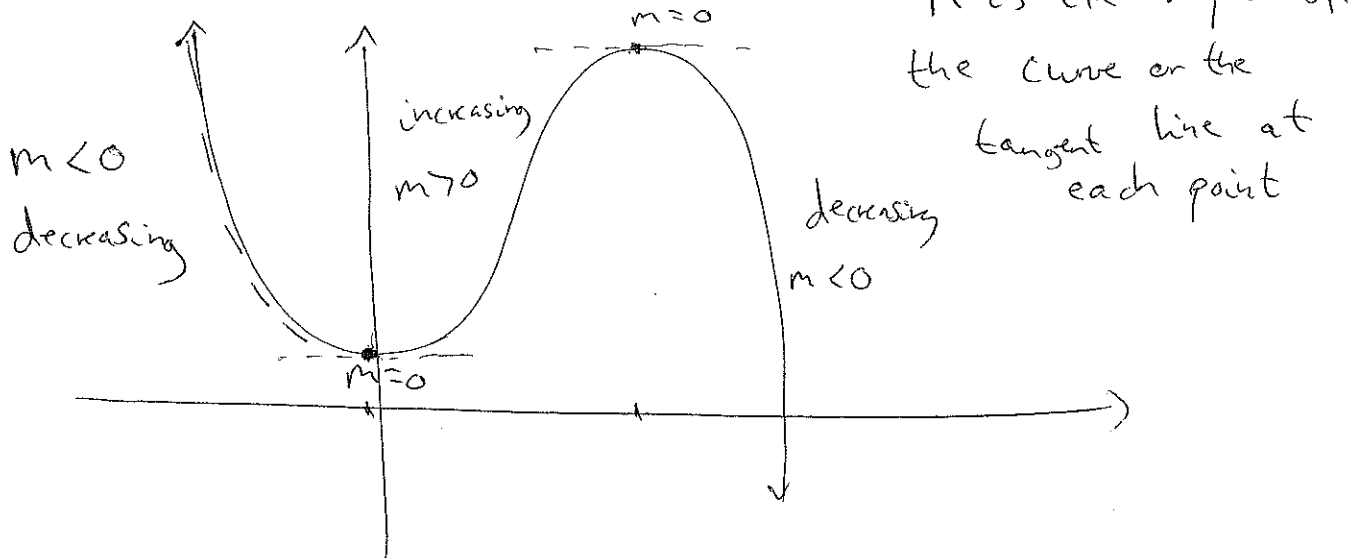
$f(x)$ is

Increasing on $(-\infty, 0)$ and $(0.6, 1.6)$

Decreasing on $(0, 0.6)$ and $(1.6, \infty)$

The slope of the tangent line is positive when the function is increasing.
 The slope of the tangent line is negative when the function is decreasing.
 The slope of the tangent line is zero at "peaks" and "valleys".

EX

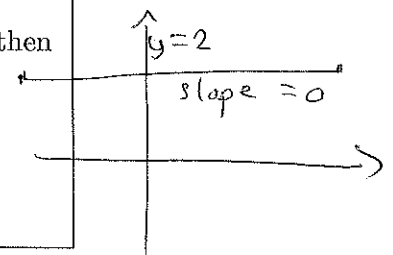


Test for intervals where $f(x)$ is increasing and decreasing:

Suppose a function f has a derivative at each point in an open interval, then

- if $f'(x) > 0$ for each x in the interval, f is **increasing** on the interval.
- if $f'(x) < 0$ for each x in the interval, f is **decreasing** on the interval.
- if $f'(x) = 0$ for each x in the interval, f is **constant** on the interval.

EX



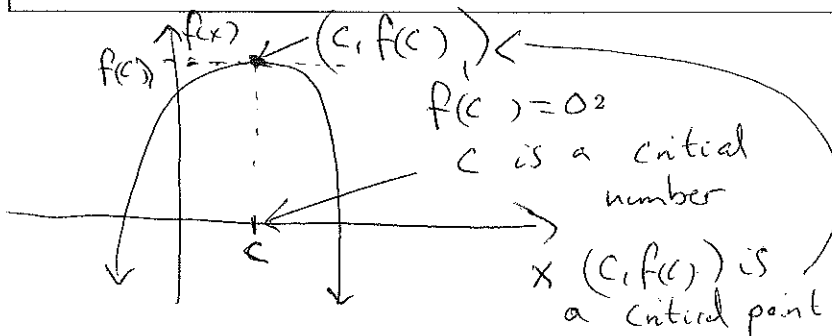
The derivative $f'(x)$ can change signs from positive to negative or from negative to positive at points where $f'(x) = 0$ and at points where $f'(x)$ does not exist. The values of x where this occurs are called **critical numbers**.

Critical numbers:

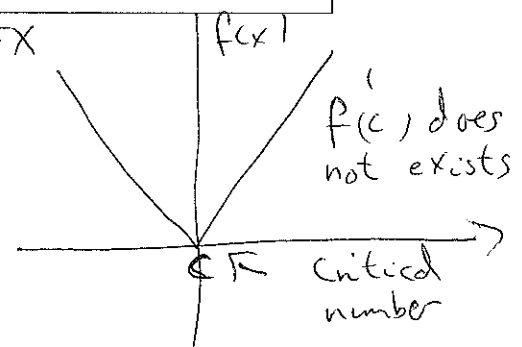
The **critical numbers** for a function f are those numbers c in the domain of f for which $f'(c) = 0$ or $f'(c)$ does not exist.

A **critical point** is a point whose x -coordinate is the critical number c and whose y -coordinate is $f(c)$, i.e. $(c, f(c))$.

EX.



EX



The following method will give an approach on how to find where a function is increasing and decreasing on an open interval:

Increasing and decreasing test:

1. Determine the critical numbers for f and put them on a number line. Determine also any points where f is undefined. These points determine several open intervals.
2. Determine whether $f'(x) > 0$ or $f'(x) < 0$ on each interval.
3. Use the test on the previous page to decide whether f is increasing or decreasing on the interval.

Example. For the following functions determine the intervals in which the function is increasing or decreasing. Locate all the points where the tangent line is horizontal.

1. $f(x) = x^3 + 3x^2 - 24x + 1$

Put $f'(x) = 0$

$$f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8)$$

$$= 3(x+4)(x-2) = 0$$

$$x+4=0 \text{ or } x-2=0$$

$$x=-4 \text{ or } x=2$$

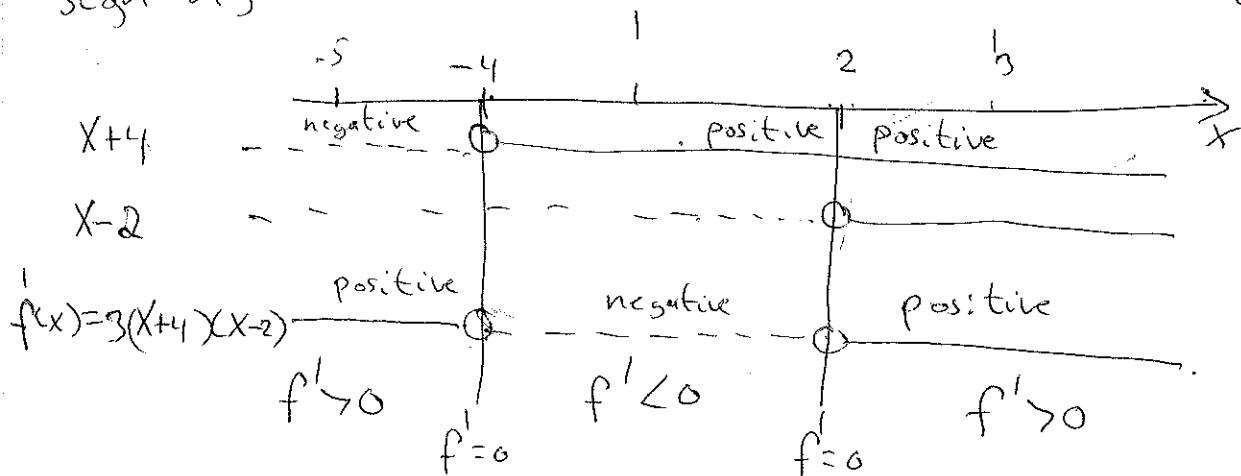
Critical numbers:

$$x = -4$$

$$x = 2$$

$f'(x)$ exists everywhere.

Sign-diagram



$f(x)$ is increasing on $(-\infty, -4)$ and $(2, \infty)$
 $f(x)$ is decreasing on $(-4, 2)$

Target line is horizontal where $f'(x) = 0$
 $x = -4$
 $x = 2$

$x+4$
 Plug in $x = -5$
 $-5 + 4 = -1$ negative
 Plug in $x = 1$
 $1 + 4 = 5$ positive
 ~~$x-2$~~
 Plug in $x = 1$
 $1 - 2 = -1$ negative
 Plug in $x = 3$
 $3 - 2 = 1$ positive

$$f(x) = x^3 + 3x^2 - 9x$$

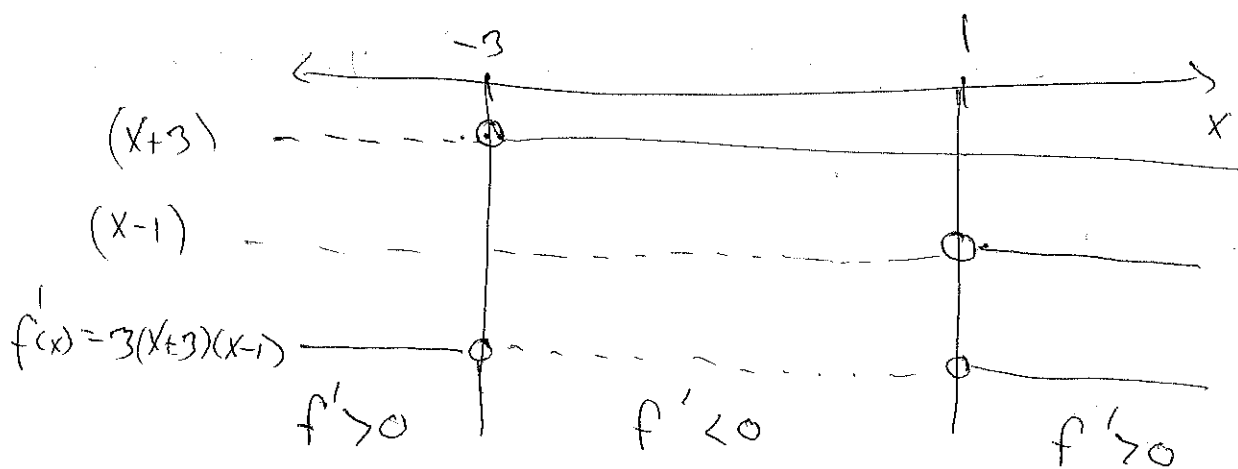
Find the critical numbers and where f is increasing and decreasing.

$$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3)$$

$$= 3(x+3)(x-1) = 0$$

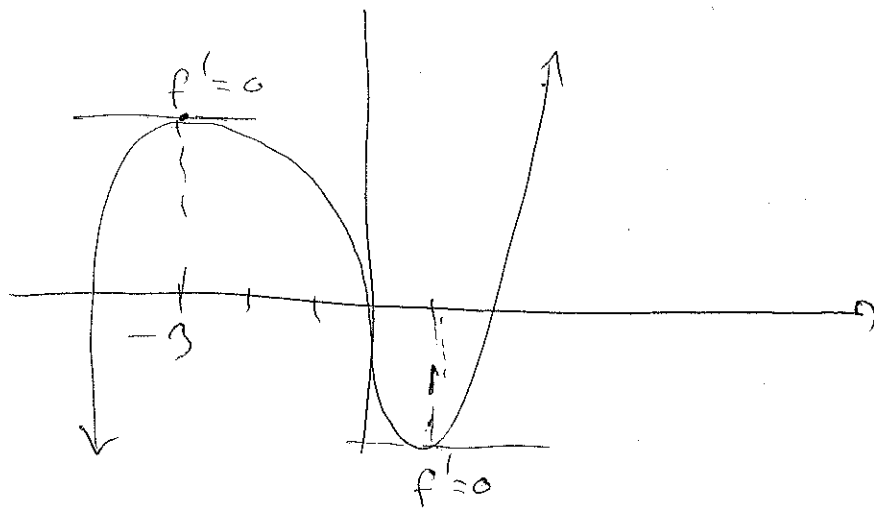
$$x+3=0 \text{ or } x-1=0$$

Critical numbers: $x = -3$ or $x = 1$



$f(x)$ is increasing on $(-\infty, -3)$ and $(1, \infty)$ $f' > 0$

$f(x)$ is decreasing on $(-3, 1)$ $f' < 0$



$$2. f(x) = \frac{x+2}{x-3}$$

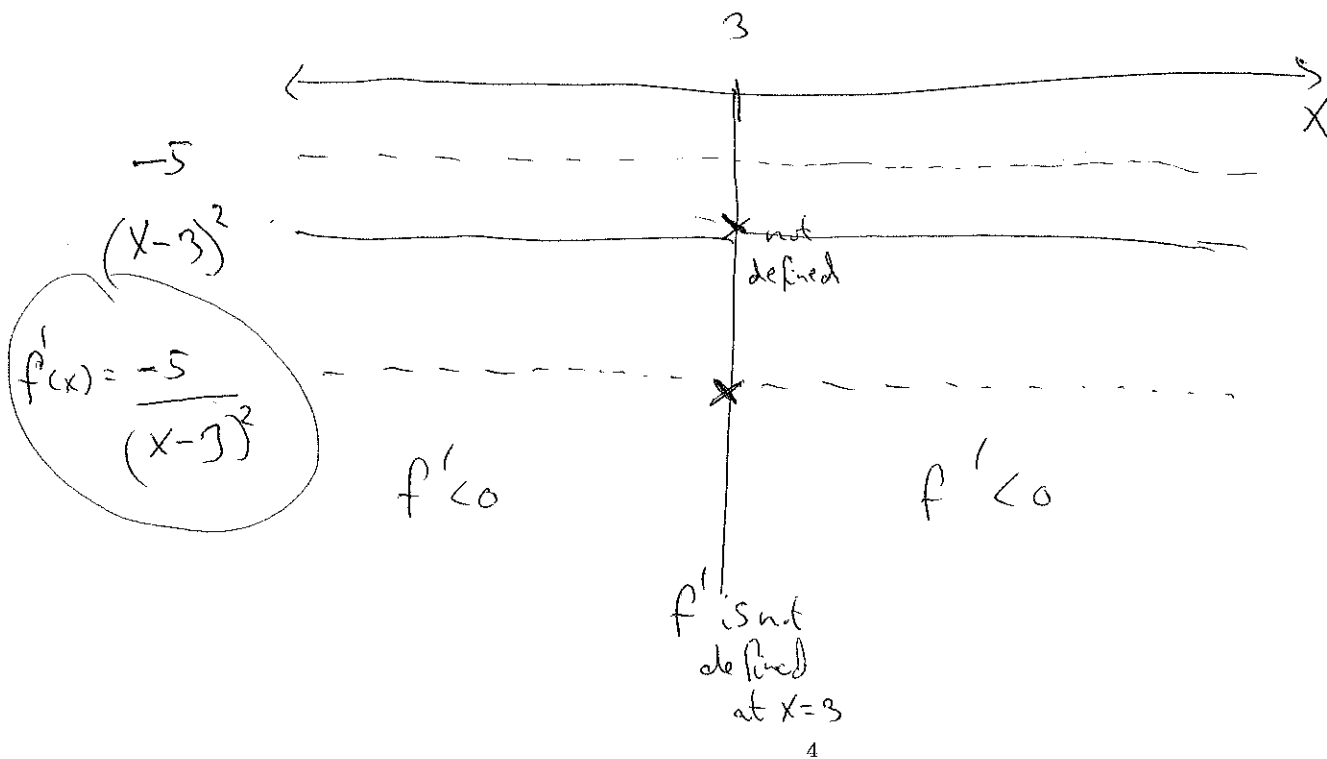
$$f'(x) = \frac{1 \cdot (x-3) - 1 \cdot (x+2)}{(x-3)^2} = \frac{-5}{(x-3)^2}$$

$$f'(x) = 0 \quad \frac{-5}{(x-3)^2} = 0 \quad \text{has no solution.}$$

$f'(x)$ is not defined for $x=3$

Since $x=3$ is not in the domain of $f(x)$, $x=3$ is not a critical number.

Hence, there are no critical numbers



$f(x)$ is decreasing on $(-\infty, 3)$ and $(3, \infty)$

$$g(x) = u(x) \cdot v(x)$$

$$g'(x) = u'(x) \cdot v(x) + v'(x) \cdot u(x)$$

$$3. g(x) = x e^{-2x^2}$$

$$g'(x)$$

$$u(x) = x$$

$$v(x) = e^{-2x^2}$$

$$u'(x) = 1$$

$$v'(x) = e^{-2x^2} \cdot (-4x)$$

$$= 1 \cdot e^{-2x^2} + (-4x) \cdot e^{-2x^2} \cdot x$$

$$= e^{-2x^2} - 4x^2 e^{-2x^2} = e^{-2x^2} (1 - 4x^2)$$

$$= e^{-2x^2} (1 - 2x)(1 + 2x) = 0$$

$$e^{-2x^2} > 0$$

for all x

$$1 - 2x = 0$$

$$1 + 2x = 0$$

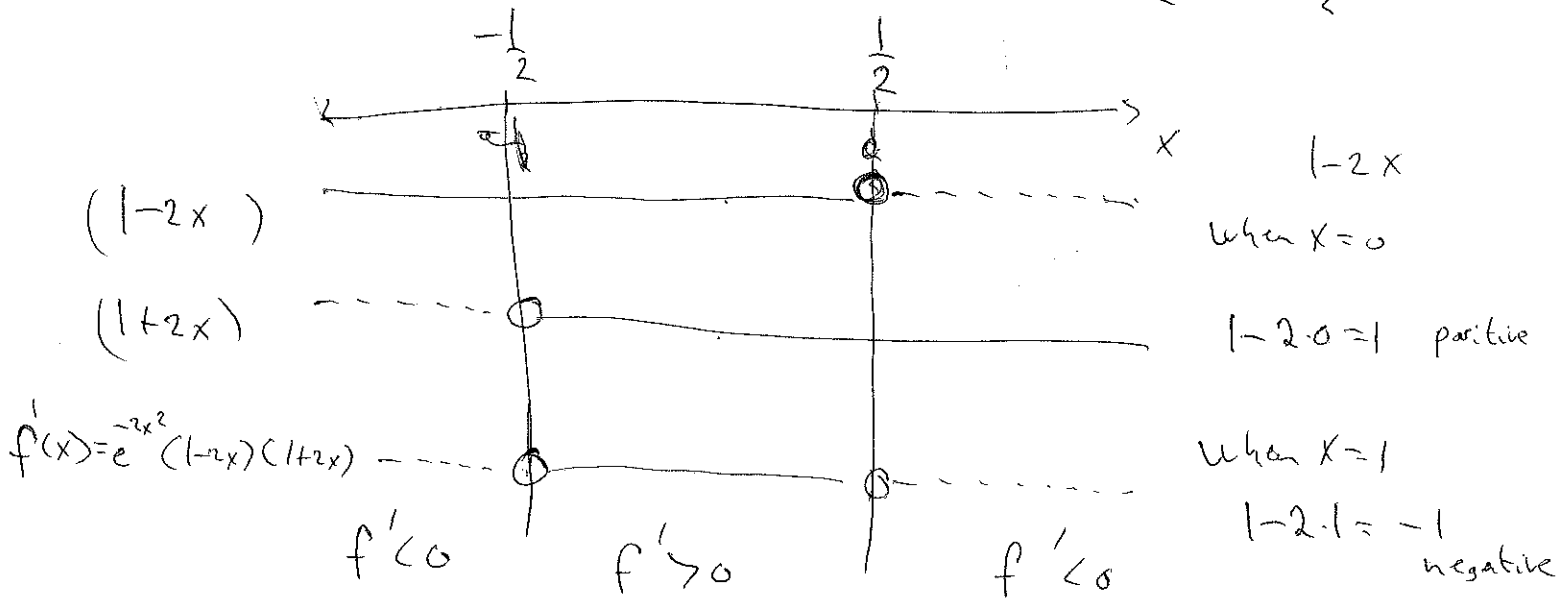
$$1 = 2x$$

$$1 = -2x$$

$$x = \frac{1}{2}$$

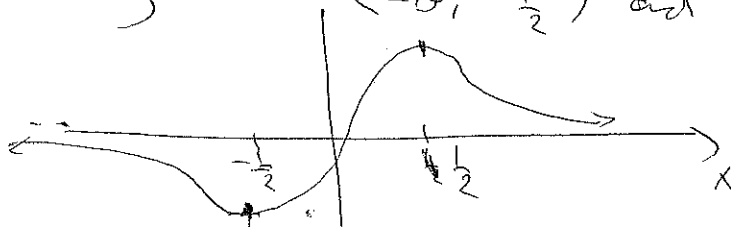
$$x = -\frac{1}{2}$$

critical numbers : $x = -\frac{1}{2}$, $x = \frac{1}{2}$



f is increasing on $(-\frac{1}{2}, \frac{1}{2})$

f is decreasing on $(-\infty, -\frac{1}{2})$ and $(\frac{1}{2}, \infty)$



Domain of f is

$$D = [-4, 4]$$

4. $f(t) = \sqrt{16 - t^2}$

$$f'(t) = \frac{1}{2\sqrt{16-t^2}} (-2t) = \frac{-t}{\sqrt{16-t^2}} = 0$$

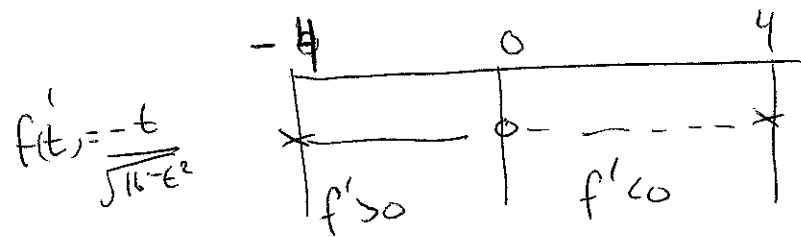
$$\Rightarrow t = 0$$

$$16 - t^2 = 0$$

$$t = \pm 4$$

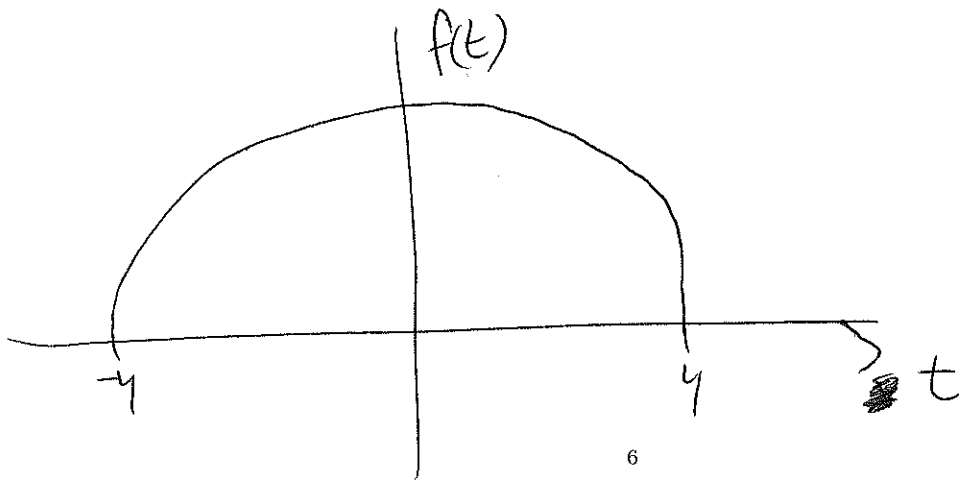
$f'(t)$ is not defined at $t = \pm 4$,
but $t = \pm 4$ is in the domain
of f .

Thus, the critical numbers are: $t = -4, 0, 4$.



f is increasing on $(-4, 0)$

f is decreasing on $(0, 4)$



Example. Suppose a company that sells TV's finds that the cost per TV decreases linearly with the number sold monthly, decreasing from \$500 when none are sold to \$250 when 500 are sold.

(A) Find the average cost function $\bar{C}(x)$, where x is the number of TV's sold monthly for $0 \leq x \leq 500$.

$(0, 500)$ $(500, 250)$ are points on $\bar{C}(x)$.

$$m = \frac{250 - 500}{500 - 0} = -0.5$$

$$\bar{C}(x) = -0.5x + 500$$

(B) Find the cost function, $C(x)$.

$$\bar{C}(x) = \frac{C(x)}{x}$$

$$\begin{aligned} \Rightarrow C(x) &= x \cdot \bar{C}(x) = x \cdot (-0.5x + 500) \\ &= -0.5x^2 + 500x \end{aligned}$$

(C) Suppose the revenue function can be approximated by

$$R(x) = 0.0001x^3 - 0.62x^2 + 535x \text{ for } 0 \leq x \leq 500.$$

Find the profit function, $P(x)$.

$$\begin{aligned} P(x) &= R(x) - C(x) = 0.0001x^3 - 0.62x^2 + 535x - (-0.5x^2 + 500x) \\ &= 0.0001x^3 - 0.12x^2 + 35x \end{aligned}$$

$$P'(x) = 0.0003 \cdot (x-608.2)(x-191.8)$$

(D) Find the intervals on which $P(x)$ is increasing and decreasing. Interpret your answer.

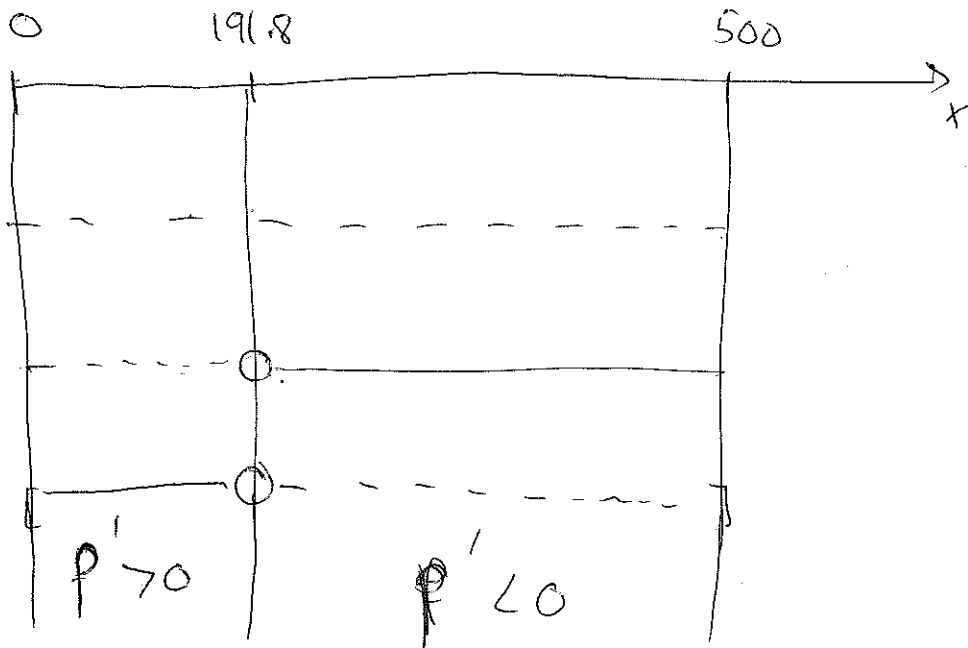
$$P(x) = 0.0001(x^3) - 0.12x^2 + 35x \quad 0 \leq x \leq 500$$

$$\begin{aligned} P'(x) &= 3 \cdot 0.0001x^2 - 0.12 \cdot 2x + 35 \\ &= 0.0003x^2 - 0.24x + 35 = 0 \end{aligned}$$

$$x = \frac{0.24 \pm \sqrt{(0.24)^2 - 4 \cdot 0.0003 \cdot 35}}{2 \cdot 0.0003}$$

$$= \frac{0.24 \pm \sqrt{0.0156}}{0.0006} = 608.2 \text{ or } 191.8$$

$$P(x) = 0.0003 \cdot (x-608.2)$$



$$x-608.2$$

$$\text{Let } x=1$$

$$1-608.2 = -607.2 < 0$$

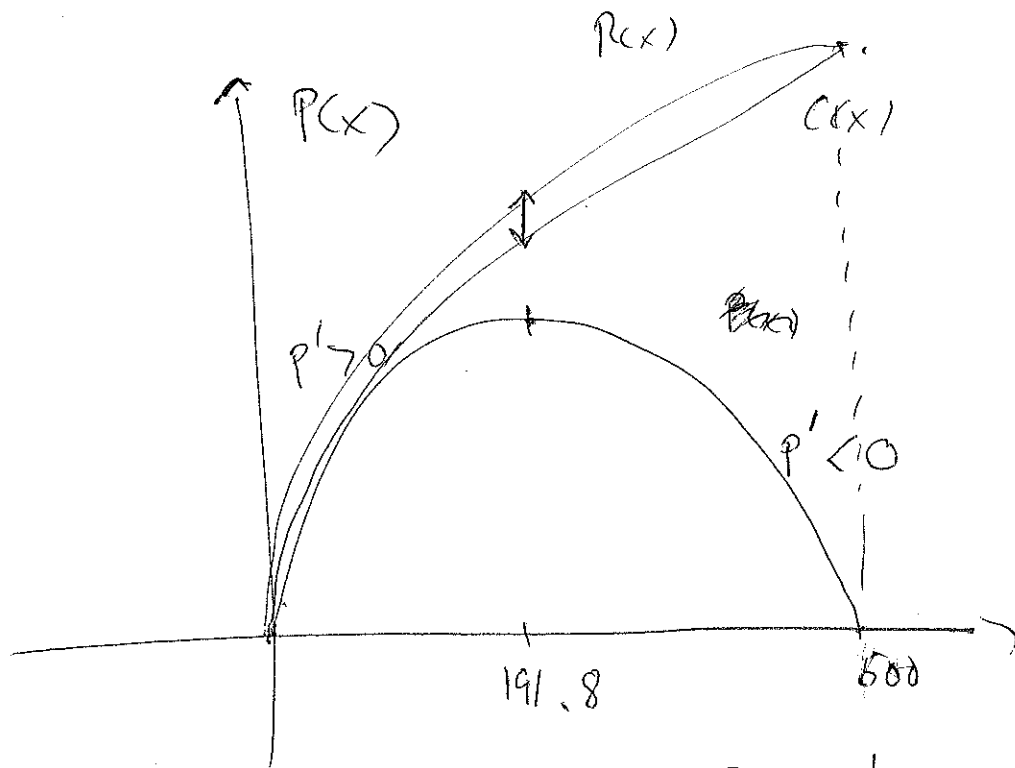
$$x-191.8$$

$$\text{Let } x=1$$

$$1-191.8$$

$$= -190.8 < 0$$

P is increasing on $(0, 191.8)$ and decreasing on $(191.8, 500)$



2. method:

Sign-diagram

Test point

Test point

	0	1	191.8	200	500
$P'(x) = 0.0003x^2 - 0.24x + 35$		+	0	-	
		$P' > 0$		$P' < 0$	

$$P'(1) = 0.0003 \cdot 1^2 - 0.24 \cdot 1 + 35$$

$$= 34.8 > 0$$

$$P'(200) = 0.0003 \cdot (200)^2 - 0.24 \cdot 200 + 35$$

$$= -1 < 0$$