4.5: Derivatives of Logarithmic Functions

In this section we will find derivatives of logarithmic functions.

Recall that the logarithmic functions and exponential functions are inverses of each other.

Also recall that $\frac{d}{dx}a^x = \ln(a)a^x$.

Derivative of $\log_a x$:

For any $a > 0$, $a \neq 1$, we have

$$\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$$

When $a = e$, we have:

Derivative of $\ln x$:

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Proof:

$$f(x) = \log_a x$$

$$a^{f(x)} = x$$

Take the derivative on both sides of the equation.

$$\ln(a) \cdot a^{f(x)} \cdot f'(x) = 1$$

$$(\ln a) \cdot x \cdot f'(x) = 1$$

$$f'(x) = \frac{1}{(\ln a) \cdot x}$$

Example. Find the derivatives of the following functions:

(A) $f(x) = \ln 5x$ $= \ln(5) + \ln(x)$

$$f'(x) = \frac{d}{dx}\left(\ln(5) + \ln(x)\right) = \frac{d}{dx}\ln(5) + \frac{d}{dx}\ln(x) = 0 + \frac{1}{x} = \frac{1}{x}$$

(B) $g(x) = \log x$ $= \log_{10}(x)$

$$g'(x) = \frac{1}{(\ln 10)x}$$

Apply the chain rule to find

(A) $\dfrac{d}{dx}\log_a g(x)$ $= \dfrac{1}{\ln a} \cdot \dfrac{g'(x)}{g(x)}$

(B) $\dfrac{d}{dx}\ln g(x)$ $= \dfrac{g'(x)}{g(x)}$

$f(x) = \ln(5x)$

use the chain rule to find

$f'(x) = \dfrac{g'(x)}{g(x)} = \dfrac{1 \cdot \cancel{5}}{\cancel{5}x} = \dfrac{1}{x}$

$g(x) = 5x$

$g'(x) = 5$

Example. Apply the chain rule to find the derivative of each function.

(A) $f(x) = \ln(x^3 + 2)$

$$g(x) = x^3 + 2$$

$$g'(x) = 3x^2$$

$$f'(x) = \frac{g'(x)}{g(x)} = \frac{3x^2}{x^3+2}$$

(B) $y = \log_3(\sqrt{4x^2 - 2x})$

$$g(x) = \sqrt{4x^2 - 2x}$$

$$g'(x) = \frac{1}{2}(4x^2-2x)^{-\frac{1}{2}}(8x-2)$$

$$\frac{dy}{dx} = \frac{1}{(\ln 3)} \cdot \frac{g'(x)}{g(x)}$$

$$= \frac{1}{(\ln 3)} \frac{\frac{1}{2}(4x^2-2x)^{-\frac{1}{2}}(8x-2)}{\sqrt{4x^2-2x}}$$

$$= \frac{1}{(\ln 3)} \cdot (4x-1) \cdot \frac{1}{\sqrt{4x^2-2x} \cdot \sqrt{4x^2-2x}}$$

$$= \frac{1}{(\ln 3)} (4x-1) \cdot \frac{1}{4x^2-2x}$$

$$= \frac{1}{(\ln 3)} \frac{4x-1}{4x^2-2x}$$

Assume $g'(x)$ exists and $g(x) \neq 0$.
For any $a > 0$, $a \neq 1$, we have

$$\frac{d}{dx}[\log_a |x|] = \frac{1}{(\ln a)x}$$
$$\frac{d}{dx}[\ln |x|] = \frac{1}{x}$$
$$\frac{d}{dx}[\log_a |g(x)|] = \frac{1}{(\ln a)}\frac{g'(x)}{g(x)}$$
$$\frac{d}{dx}[\ln |g(x)|] = \frac{g'(x)}{g(x)}$$

Example. Find the derivative of the function

$$y = \ln |3x| \qquad g(x) = 3x \quad g'(x) = 3$$

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{3}{3x} = \underline{\underline{\frac{1}{x}}}$$

Example. Suppose the demand function for q units of a certain item is

$$p = D(q) = 150 + \frac{30}{\ln q} \quad q > 1,$$

where p is in dollars.

(A) Find the marginal revenue.

Revenue:

$$R(q) = q \cdot p = q \cdot \left(150 + \frac{30}{\ln q}\right)$$

$$= 150q + \frac{30q}{\ln(q)}$$

Marginal revenue:

$$R'(q) = 150 + \frac{30 \cdot \ln(q) - \frac{1}{\cancel{q}} \cdot 30\cancel{q}}{(\ln q)^2} = 150 + \frac{30\ln(q) - 30}{(\ln q)^2}$$

$$= 150 + \frac{30(\ln q - 1)}{(\ln q)^2}$$

(B) Approximate the revenue from one more unit when 7 units are sold.

The revenue from one more unit is

$$\frac{dR}{dq}$$

For $q = 7$,

$$R'(7) = 150 + \frac{30(\ln(7) - 1)}{(\ln 7)^2}$$

$$= 157.49$$

The revenue from one more unit when 7 units are sold is $\underline{\underline{\$\,157.49}}$