4.4: Derivatives of Exponential Functions

In this section we will find derivatives of exponential functions.

Derivative of e^x:

$$\frac{d}{dx}(e^x) = e^x$$

Derivative of a^x:

For any positive constant $a \neq 1$, we have

$$\frac{d}{dx}(a^x) = (\ln a)a^x.$$

The exponential function with base e has the simplest derivatives of all exponential functions.

Example. Let $f(x) = 3 \cdot 2^x$. Find $f'(x)$

$$f'(x) = 3 \cdot \ln(2) \cdot 2^x$$

We use the chain rule to prove the next two formulas:

Derivative of $a^{g(x)}$ and $e^{g(x)}$:

$$\frac{d}{dx}(a^{g(x)}) = (\ln a)a^{g(x)}g'(x)$$

and

$$\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$$

Example. Find the derivative of each of the following functions

(A) $g(x) = e^{4x}$

$f(x) = 4x$

$f'(x) = 4$

$g'(x) = e^{f(x)} \cdot f'(x)$

$= e^{4x} \cdot 4$

(B) $y = 3 \cdot 5^{x^2+1}$

$\frac{d}{dx} x^2+1 = 2x$

$\frac{dy}{dx} = 3 \cdot 2x \cdot 5^{x^2+1}$

$= 6x \cdot 5^{x^2+1}$

(C) $g(x) = e^{\sqrt{x^3-1}}$.

$g'(x) = \frac{1}{2\sqrt{x^3-1}} \cdot 3x^2 \cdot e^{\sqrt{x^3-1}}$

$\frac{d}{dx} \sqrt{x^3-1} = \frac{1}{2}(x^3-1)^{-\frac{1}{2}} \cdot 3x^2$

$= \frac{1}{2\sqrt{x^3-1}} \cdot 3x^2$

Example. The sales, $S(t)$, in dollars of a certain product as a function of time, t, in years is given by the formula

$$S(t) = \frac{2000}{1 + 19e^{-0.5t}}.$$

(A) What is the initial sale? (The sales at time $t = 0$).

$$S(0) = \frac{2000}{1+19e^{-0.5\cdot 0}} = \frac{2000}{1+19} = \frac{2000}{20} = 100$$

Initial Sale is $\underline{\$ 100}$

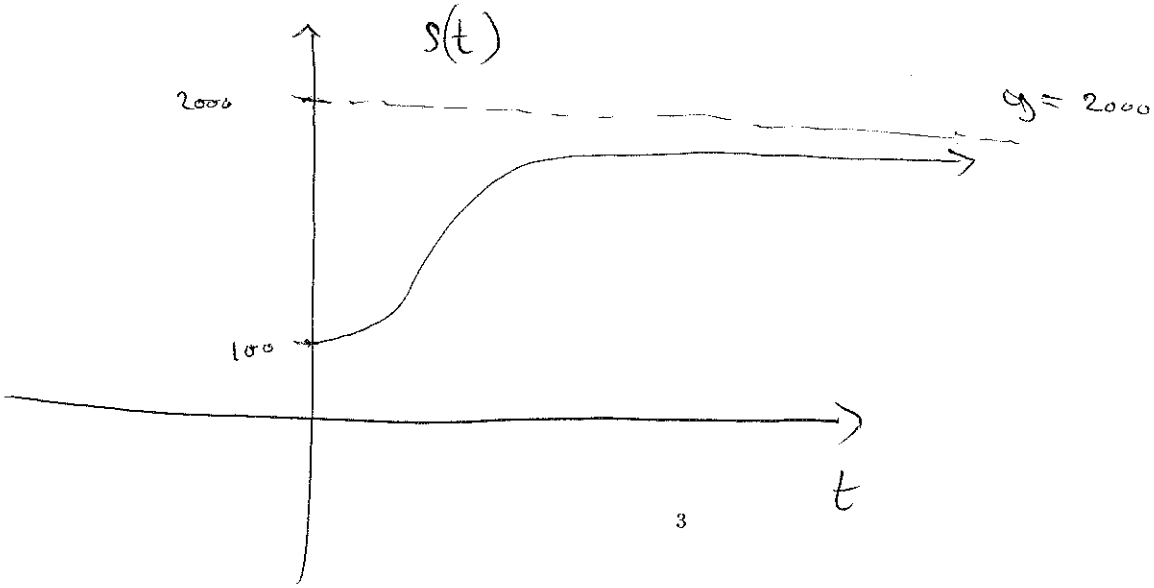
(B) What is happening to the sales as time goes on?

$$\lim_{t\to\infty} S(t) = \lim_{t\to\infty} \frac{2000}{1+19e^{-0.5t}} = \frac{2000}{1+19\cdot 0} = 2000$$

Since $\lim_{t\to\infty} e^{-0.5t} = 0$ Sales approaches $\underline{\$ 2000}$

(C) Find the rate of change of sales at each time.

$$S'(t) = \frac{-19(-0.5)e^{-0.5t}\cdot 2000}{\left(1+19e^{-0.5t}\right)^2} = \frac{-19000e^{-0.5t}}{\left(1+19e^{-0.5t}\right)^2}$$



(D) Find the rate of change of sales after 3 years. Interpret your answer.

$$S'(3) = \frac{-19000\, e^{-0.5 \cdot 3}}{\left(1+19\, e^{-0.5 \cdot 3}\right)^2} \approx 154.43$$

The rate of change of sales after 3 years is \$154.43 / year.
Thus, after 3 years, the sales increases by \$154.43 per year.

(E) What is happening to the rate of change of sales as time goes on?

$$\lim_{t \to \infty} S'(t) = \lim_{t \to \infty} \frac{-19000\, e^{-0.5t}}{\left(1+19\, e^{-0.5t}\right)^2} = \frac{0}{(1+0)^2} = 0$$

The rate of change of sales approaches 0 as time goes on.