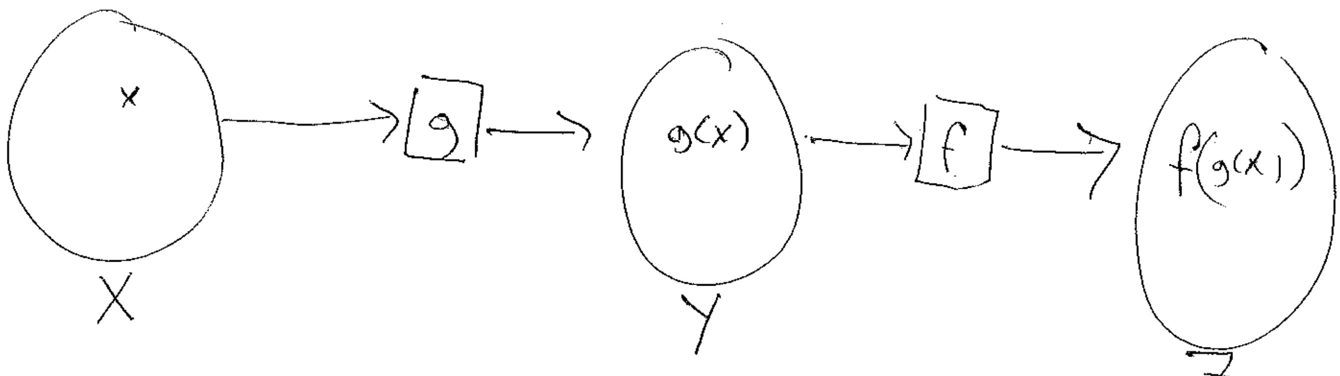
4.3: The Chain Rule

In this section we will consider compositions of functions. We will then find derivatives of compositions of functions by using the chain rule.

Composite function:

Let f and g be functions. The **composite function**, or **composition**, of f and g is the function whose values are given by $f[g(x)]$ for all x in the domain of g such that $g(x)$ is in the domain of f.

$f[g(x)]$ is read as "f of g of x".



Example. Let $f(x) = 3x - 2$ and $g(x) = \sqrt{2x+1}$.

(A) Find $f[g(5)]$ $\qquad g(5) = \sqrt{2 \cdot 5 + 1} = \sqrt{11}$

$= f(\sqrt{11}) = 3 \cdot \sqrt{11} - 2$

(B) Find $f[g(x)] = f\left(\sqrt{2x+1}\right) = 3\left(\sqrt{2x+1}\right) - 2$

Example. Suppose the quantity, q, demanded as a function of time, t, in hours is given by $q(t) = 2t$. Suppose the cost, $C(q)$, in dollars to produce q items is given by $C(q) = 5q + 3$. Find the rate of change of cost with respect to time t, i.e. find $\frac{dC}{dt}$

$q(t) = 2t$

$\frac{dq}{dt} = 2$ q increases by 2 items demanded per hour = 2 items / hour

$C(q) = 5q + 3$

$\frac{dC}{dq} = 5$ C increases by 5 dollar per item demanded = 5 dollar / item

$\frac{dC}{dt}$ = 5 dollar / ~~item~~ · 2 ~~items~~ / hour

= 10 dollar / hour

units \$ / hour

$\frac{\text{dollar}}{\cancel{\text{item}}} \cdot \frac{\cancel{\text{item}}}{\text{hour}}$

$$\frac{dC}{dt} = \frac{dC}{dq} \cdot \frac{dq}{dt}$$

Chain Rule:

If y is a function of u, say $y = f(u)$, and u is a function of x, say $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example. Let $y = (3x+1)^4$. Use the chain rule to find $\frac{dy}{dx}$.

$$y = f(u) = u^4 \qquad \frac{dy}{du} = 4u^3$$

$$u = g(x) = 3X+1 \qquad \frac{du}{dX} = 3$$

$$y = f(u) = f(g(x)) = f(3X+1) = (3X+1)^4$$

chain rule

$$\frac{dy}{dX} = \frac{dy}{du} \cdot \frac{du}{dX} = 4u^3 \cdot 3 = 4(3X+1)^3 \cdot 3$$

$$= 12 \cdot (3X+1)^3$$

Chain Rule (Alternate form):

If $y = f[g(x)]$, then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

To find the derivative of $f[g(x)]$, find the derivative of $f(x)$, replace each x with $g(x)$, and then multiply the result by the derivative of $g(x)$.

Example. Let $y = \sqrt{6x-4}$. Use the chain rule to find $\frac{dy}{dx}$.

outside function : $f(x) = \sqrt{X} = X^{\frac{1}{2}}$

Inside function : $g(x) = 6X-4$

$$y = f(g(x)) = f(6X-4)$$

$$= \sqrt{6X-4}$$

$$f'(x) = \frac{1}{2} X^{-\frac{1}{2}}$$

$$g'(x) = 6$$

$$f'(g(x)) = f'(6X-4) = \frac{1}{2}(6X-4)^{-\frac{1}{2}}$$

$$\frac{dy}{dX} = f'(g(x)) \cdot g'(x) = \frac{1}{2}(6X-4)^{-\frac{1}{2}} \cdot 6 = 3(6X-4)^{-\frac{1}{2}} = \frac{3}{\sqrt{6X-4}}$$

Example. Use the chain rule to find $D_x(x^2+4x)^5$

outside function $f(x) = x^5$ | $f'(x) = 5x^4$

Inside function: $g(x) = x^2+4x$ | $g'(x) = 2x+4$

$f(g(x)) = (x^2+4x)^5$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$= 5(x^2+4x)^4 \cdot (2x+4)$$

$f'(g(x))$

$= f'(x^2+4x)$

$= 5(x^2+4x)^4$

Example. Let $y = \frac{2x+5}{(x^2-3)^3}$. Find $\frac{dy}{dx}$

$$y = f(g(x))$$

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

$$f(x) = \frac{u(x)}{v(x)}$$

$$f'(x) = \frac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{[v(x)]^2}$$

$u(x) = 2x+5$

$v(x) = (x^2-3)^3$

outside function: $f(x) = (x)^3$

Inside function: $g(x) = x^2 - 3$

$f'(x) = 3x^2$

$g'(x) = 2x$

$f'(g(x)) = f'(x^2-3) = 3(x^2-3)^2$

$\boxed{v'(x) = 3(x^2-3)^2 \cdot 2x}$

$= 6x(x^2-3)^2$

$u'(x) = 2$

$v'(x) =$

$$f'(x) = \frac{2 \cdot (x^2-3)^3 - 6x(x^2-3)^2(2x+5)}{((x^2-3)^3)^2}$$

$$= \frac{2(x^2-3)^3 - 6x(x^2-3)^2(2x+5)}{(x^2-3)^6}$$

$$= \frac{\cancel{(x^2-3)^2} \cdot [2(x^2-3)^1 - 6x(2x+5)]}{\cancel{(x^2-3)^2}\ (x^2-3)^4}$$

$$= \frac{2x^2 - 6 - 12x^2 - 30x}{(x^2-3)^4} = \frac{-10x^2 - 30x - 6}{(x^2-3)^4}$$

$A = 1000\left(1 + \frac{r}{12}\right)^{12 \cdot 5}$

$\frac{r}{100}$

Example. Suppose a sum of \$1000 is deposited in an account with an interest rate of r percent per year compounded monthly. At the end of 5 years, the balance in the account is

$$A = 1000(1 + \frac{r}{1200})^{60}.$$

Find the rate of change of A with respect to r if $r = 3\%$.

$$A'(r) = \frac{dA}{dr} = 1000 \cdot \frac{d}{dr}\left(1 + \frac{r}{1200}\right)^{60}$$

$$= 1000 \cdot 60\left(1 + \frac{r}{1200}\right)^{59} \cdot \frac{1}{1200}$$

$$= 50\left(1 + \frac{r}{1200}\right)^{59}$$

$$A'(3) = 50\left(1 + \frac{3}{1200}\right)^{59}$$

$$\approx 57.94$$

Chain rule:

outside fct: $f(r) = r^{60}$

Inside fct: $g(r) = 1 + \frac{r}{1200}$

$f'(r) = 60 r^{59}$

$g'(r) = \frac{1}{1200}$

$f'(g(r)) = f'\left(1 + \frac{r}{1200}\right)$

$= 60\left(1 + \frac{r}{1200}\right)^{59}$

A is increasing by \$57.94 per percentage point if $r = 3\%$

Example. Suppose the cost in dollars of producing q items is given by

$$C(q) = 1000q + 200$$

and the quantity demanded is given by

$$q = \sqrt{10000 - 2p},$$

where p is the price per item.

(A) Find the expression for the revenue, R, as a function of quantity demanded.

$$R(q) = p \cdot q$$

$$= \left(\frac{10000 - q^2}{2}\right) \cdot q$$

$$R(q) = 5000q - \frac{q^3}{2}$$

$$q = \sqrt{10000 - 2p}$$

$$q^2 = 10000 - 2p$$

$$2p = 10000 - q^2$$

$$p = \frac{10000 - q^2}{2}$$

(B) Find the expression for the profit, P.

$$P(q) = R(q) - C(q)$$

$$= 5000q - \frac{q^3}{2} - (1000q + 200)$$

$$= 4000q - \frac{q^3}{2} - 200$$

$$P(q) = 4000q - \frac{q^3}{2} - 200$$

(C) Find the expression for the marginal profit.

$$P'(q) = 4000 - \frac{1}{2} \cdot 3 \cdot q^2$$

(D) Determine the value of the marginal profit when the price is $2000.

$p = 2000$

$$\boxed{q = \sqrt{10000 - 2p}}$$

The quantity when $p = 2000$ is

$$q = \sqrt{10000 - 2 \cdot 2000} = \sqrt{6000}$$

$$q^2 = 6000$$

$P'(q)$

$$P'(\sqrt{6000}) = 4000 - \frac{1}{2} \cdot 3 \cdot 6000$$

$$= -5000$$

when the price is $2000, the profit decreases by $5000 per dollar increase in price