

4.3: The Chain Rule

In this section we will consider compositions of functions. We will then find derivatives of compositions of functions by using the chain rule.

Composite function:

Let f and g be functions. The **composite function**, or **composition**, of f and g is the function whose values are given by $f[g(x)]$ for all x in the domain of g such that $g(x)$ is in the domain of f .

$f[g(x)]$ is read as " f of g of x ".

Example. Let $f(x) = 3x - 2$ and $g(x) = \sqrt{2x + 1}$.

(A) Find $f[g(5)]$

(B) Find $f[g(x)]$

Example. Suppose the quantity, q , demanded as a function of time, t , in hours is given by $q(t) = 2t$. Suppose the cost, $C(q)$, in dollars to produce q items is given by $C(q) = 5q + 3$. Find the rate of change of cost with respect to time t , i.e. find $\frac{dC}{dt}$

Chain Rule:

If y is a function of u , say $y = f(u)$, and u is a function of x , say $u = g(x)$, then $y = f(u) = f[g(x)]$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example. Let $y = (3x + 1)^4$. Use the chain rule to find $\frac{dy}{dx}$.

Chain Rule (Alternate form):

If $y = f[g(x)]$, then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

To find the derivative of $f[g(x)]$, find the derivative of $f(x)$, replace each x with $g(x)$, and then multiply the result by the derivative of $g(x)$.

Example. Let $y = \sqrt{6x - 4}$. Use the chain rule to find $\frac{dy}{dx}$.

Example. Use the chain rule to find $D_x(x^2 + 4x)^5$

Example. Let $y = \frac{2x+5}{(x^2-3)^3}$. Find $\frac{dy}{dx}$

Example. Suppose a sum of \$1000 is deposited in an account with an interest rate of r percent per year compounded monthly. At the end of 5 years, the balance in the account is

$$A = 1000\left(1 + \frac{r}{1200}\right)^{60}.$$

Find the rate of change of A with respect to r if $r = 3\%$.

Example. Suppose the cost in dollars of producing q items is given by

$$C(q) = 1000q + 200$$

and the quantity demanded is given by

$$q = \sqrt{10000 - 2p},$$

where p is the price per item.

(A) Find the expression for the revenue, R , as a function of quantity demanded.

(B) Find the expression for the profit, P .

(C) Find the expression for the marginal profit.

(D) Determine the value of the marginal profit when the price is \$2000.