4.2: Derivatives of products and quotients

In this section we will provide formulas for the derivatives of product and derivatives of quotients of functions.

The Product Rule:

Let $f(x) = u(x) \cdot v(x)$ and assume $u'(x)$ and $v'(x)$ both exists. Then

$$f'(x) = u'(x) \cdot v(x) + v'(x) \cdot u(x).$$

The Quotient Rule:

Let

$$f(x) = \frac{u(x)}{v(x)} \text{ and assume } u'(x) \text{ and } v'(x) \text{ both exists and } v(x) \neq 0.$$

Then

$$f'(x) = \frac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{[v(x)]^2}.$$

Example. Let $f(x) = (3x + 4)(4x^2)$. Use the product rule to find $f'(x)$

$u(x) = 3x + 4 \qquad u'(x) = 3$

$v(x) = 4x^2 \qquad v'(x) = 4 \cdot 2x = 8x$

$$f'(x) = u'(x) \cdot v(x) + v'(x) \cdot u(x)$$

$$= 3 \cdot 4x^2 + 8x \cdot (3x+4)$$

$$= 12x^2 + 24x^2 + 32x = \underline{\underline{36x^2 + 32x}}$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

Example. Let $f(x) = (\sqrt{x} - 2)(5x^3 + x)$. Use the product rule to find $f'(x)$

$u(x) = \sqrt{x} - 2 \qquad u'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

$v(x) = 5x^3 + x \qquad v'(x) = 15x^2 + 1$

$$f'(x) = u'(x) \cdot v(x) + v'(x) \cdot u(x)$$

$$= \frac{1}{2} x^{-\frac{1}{2}} \cdot (5x^3 + x) + (15x^2 + 1)(\sqrt{x} - 2)$$

Example. Let $f(x) = \frac{x^2+3x+2}{7x-1}$. Use the quotient rule to find $f'(x)$

$u(x) = x^2 + 3x + 2 \quad | \quad u'(x) = 2x + 3$

$v(x) = 7x - 1 \quad | \quad v'(x) = 7$

$$f'(x) = \frac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{[v(x)]^2}$$

$$= \frac{(2x+3)(7x-1) - 7(x^2 + 3x + 2)}{(7x-1)^2}$$

$$= \frac{14x^2 - 2x + \cancel{21x} - 3 - \cancel{7x^2} - \cancel{21x} - 14}{(7x-1)^2}$$

$$= \frac{7x^2 - 2x - 17}{(7x-1)^2}$$

Example. Let $f(x) = \frac{(x-1)(2x+4)}{x^2+3}$. Use the quotient rule to find $f'(x)$

$$u(x) = (x-1)(2x+4) = 2x^2 + 4x - 2x - 4 = 2x^2 + 2x - 4$$

$$v(x) = x^2 + 3$$

$$u'(x) = 4x + 2$$

$$v'(x) = 2x$$

$$f'(x) = \frac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{[v(x)]^2}$$

$$= \frac{(4x+2)(x^2+3) - 2x(2x^2+2x-4)}{(x^2+3)^2}$$

$$= \frac{\cancel{4x^3} + \cancel{12x} + \cancel{2x^2} + 6 - \cancel{4x^3} - \cancel{4x^2} + \cancel{8x}}{(x^2+3)^2}$$

$$= \frac{-2x^2 + 20x + 6}{(x^2+3)^2}$$

The marginal average cost:

If the total cost to manufacture x items is given by $C(x)$, then the average cost per item is $\overline{C}(x) = \frac{C(x)}{x}$. The **marginal average cost** is the derivative of the average cost function, $\overline{C'}(x)$.

Thus, the **marginal average cost** is the rate of change of average cost.

Similarly, the **marginal average revenue function**, $\overline{R'}(x)$, is defined as the derivative of the average revenue function, $\overline{R}(x) = \frac{R(x)}{x}$.

The **marginal average profit function**, $\overline{P'}(x)$, is defined as the derivative of the average profit function, $\overline{P}(x) = \frac{P(x)}{x}$.

Example. Suppose the total cost (in hundreds of dollars) to produce x units of soap is

$$C(x) = \frac{4x+3}{x+5}.$$

(A) Find the average cost to produce x units

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{\frac{4x+3}{x+5}}{x} = \frac{4x+3}{(x+5)} \cdot \frac{1}{x} = \frac{4x+3}{x^2+5x}$$

(B) Find the average cost to produce 10 units

$$\overline{C}(10) = \frac{4 \cdot 10 + 3}{10^2 + 5 \cdot 10} = \frac{43}{150}$$

Average cost to produce 10 units is \$ $\frac{4300}{150}$ per item $\approx$ \$28.67 per item

(C) Find the marginal average cost function.

$u(x) = 4x+3 \qquad u'(x) = 4$

$v(x) = x^2+5x \qquad v'(x) = 2x+5$

$$\overline{C}'(x) = \frac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{(v(x))^2}$$

$$= \frac{4 \cdot (x^2+5x) - (2x+5)(4x+3)}{(x^2+5x)^2}$$

$$= \frac{4x^2 + \cancel{20x} - 8x^2 - 6x - \cancel{20x} - 15}{(x^2+5x)^2} = \frac{-4x^2 - 6x - 15}{(x^2+5x)^2}$$

Example. Given the following table:

x	1	2
$g(x)$	3	5
$h(x)$	-1	2
$g'(x)$	-2	4
$h'(x)$	7	1

(A) Let $f(x) = 2g(x) + 3h(x)$. Find $f'(2)$

$$f'(x) = 2 \cdot g'(x) + 3 \cdot h'(x)$$

$$f'(2) = 2 \cdot g'(2) + 3 \cdot h'(2)$$

$$= 2 \cdot 4 + 3 \cdot 1 = \underline{\underline{11}}$$

(B) Let $f(x) = \frac{g(x)}{h(x)}$. Find $f'(1)$

$$f'(x) = \frac{g'(x) \cdot h(x) - h'(x) \cdot g(x)}{(h(x))^2}$$ quotient rule

$$f'(1) = \frac{g'(1) \cdot h(1) - h'(1) \cdot g(1)}{(h(1))^2}$$

$$= \frac{-2 \cdot (-1) - 7 \cdot 3}{(-1)^2} = \underline{\underline{-19}}$$