4.1: Techniques for finding the derivatives

In this section we will provide formulas for the derivatives of polynomials.
We will use the definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to develop the rules for finding derivatives.

Notations for the derivatives of $y = f(x)$:

We can write the derivative of $y = f(x)$ as

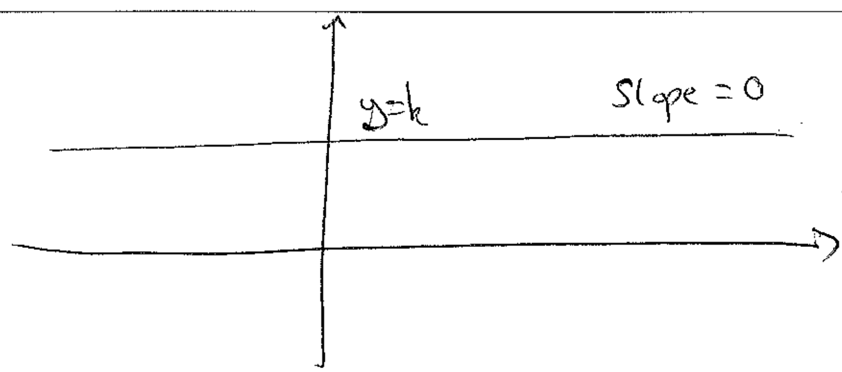
$$f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)] \quad \text{or} \quad D_x[f(x)].$$

$\dfrac{dy}{dx}$ is the derivative of y with respect to x.

Constant rule:

If $f(x) = k$, where k is any real number, then

$$f'(x) = 0.$$



Power rule:

If $f(x) = x^n$ for any real number n, then

$$f'(x) = nx^{n-1}.$$

Example. Let $f(x) = 3$. Find $f'(x)$

$f'(x) = 0$

Example. Let $y = x^3$. Find $\frac{dy}{dx}$ $n=3$

$$\frac{dy}{dx} = 3x^{3-1} = 3x^2$$

Example. Let $y = \frac{1}{t^4}$. Find $\frac{dy}{dt}$ $y = t^{-4}$

$$\frac{dy}{dt} = -4t^{-4-1} = -4t^{-5}$$

Example. Let $g(z) = \sqrt{z}$. Find $\frac{d}{dz}g(z)$

$$g(z) = z^{\frac{1}{2}}$$

$$\frac{d}{dz}g(z) = \frac{1}{2}z^{\frac{1}{2}-1} = \frac{1}{2}z^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{z}}$$

Let k be a real number and suppose $g'(x)$ exists. Let $f(x) = kg(x)$. Then
$$f'(x) = kg'(x).$$

Example. Let $g(t) = -\frac{3}{2}\frac{1}{\sqrt{t}}$. Find $g'(t)$.

$$g(t) = -\frac{3}{2}t^{-\frac{1}{2}}$$

$$g'(t) = -\frac{3}{2}\left(-\frac{1}{2}t^{-\frac{1}{2}-1}\right)$$

$$= \frac{3}{4}t^{-\frac{3}{2}}$$

Sum or difference rule:

Suppose $u'(x)$ and $v'(x)$ exists. Let $f(x) = u(x) \pm v(x)$. Then
$$f'(x) = u'(x) \pm v'(x).$$

Example. Let $y = 4x^7 - \frac{2}{x}$. Find $\frac{dy}{dx}$

$y = 4x^7 - 2x^{-1}$

$$\frac{dy}{dx} = 4 \cdot 7x^{7-1} - 2(-x^{-1-1})$$
$$= 28x^6 + 2x^{-2} = 28x^6 + \frac{2}{x^2}$$

Example. Let $f(x) = \frac{2\sqrt{x}+x^2}{x}$. Find $D_x[f(x)]$

$$f(x) = \frac{2x^{\frac{1}{2}} + x^2}{x} = 2x^{\frac{1}{2}-1} + x$$
$$= 2x^{-\frac{1}{2}} + x$$

$$D_x[f(x)] = 2\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right) + 1 \cdot x^{1-1}$$
$$= -x^{-\frac{3}{2}} + 1$$

Marginal analysis:

The word marginal refers to rates of changes.

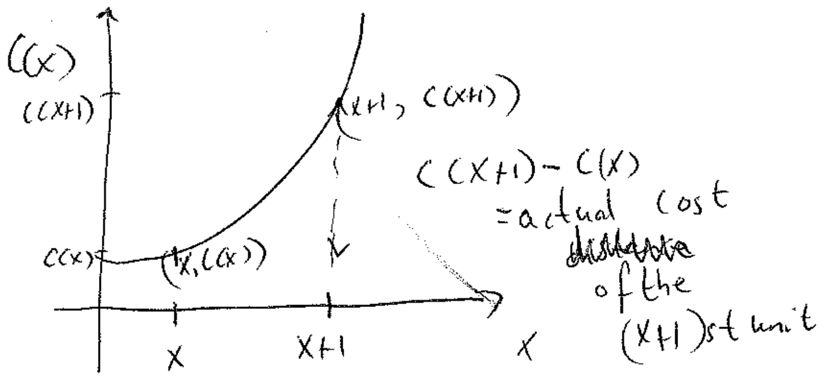
For example, marginal cost, refers to rate of change of the cost function with respect to the production at a given production level, x.

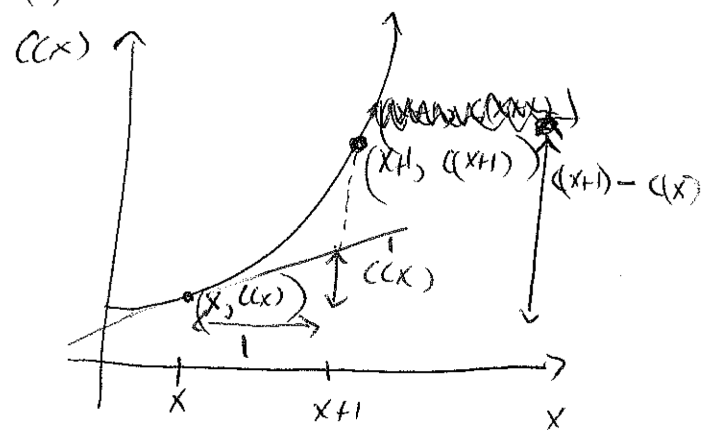
Recall that the marginal cost (or revenue, profit) is found by taking the derivatives of the cost function (or revenue, profit).

The marginal cost at some level of production x is the cost to produce the $(x+1)st$ item.

$C(x)$ represents the cost of producing x units of some item.
$C(x+1)$ represents the cost of producing $x+1$ units of some item.

The cost of the $(x+1)st$ unit is therefore $C(x+1) - C(x)$.





$C'(x)$ represents the slope of the tangent line at any point $(x, C(x))$.

Slope of tangent line: $\dfrac{\Delta y}{\Delta x} = C'(x) = \dfrac{C'(x)}{1}$

$C'(x)$ is very close to $C(x+1) - C(x)$. The two values are closest when x is large.

The demand function, defined by $p = D(q)$, is the number of units q of an item demanded when the price is p per item.
The total revenue, $R(q)$, is the product of the price per item and the amount demanded (or sold) and it is given by,

$$R(q) = q \cdot p = q \cdot D(q).$$

Example. (A) Suppose the price in dollars of a stereo system is given by

$$p(q) = \frac{500}{q^2} + 500,$$

where q represents the demand for the product. Find the marginal revenue when the demand is 5.

$$R(q) = q \cdot p = q \cdot \left(\frac{500}{q^2} + 500\right) = \frac{500}{q} + 500q$$

q is the variable

$$R'(q) = 500 \cdot (-1\, q^{-1-1}) + 500 \cdot 1 \qquad = 500\, q^{-1} + 500q$$

$$= -500\, q^{-2} + 500$$

$$R'(5) = -500\, 5^{-2} + 500 = \frac{-500}{25} + 500 = 480$$

The marginal revenue when the demand is 5 units is \$480.
~~The revenues~~ When the demand is 5 units, the revenue is increasing by \$480 per item demanded.

(B) Suppose the cost in dollars of producing q stereo systems is given by

$$C(q) = 0.5q^2 + 5q + 40.$$

Find the marginal profit when the demand is 5. Interpret your answer.

$$P(q) = R(q) - C(q) = 500\, q^{-1} + 500q - (0.5\, q^2 + 5q + 40)$$

$$P(q) = 500\, q^{-1} - 0.5\, q^2 + 495q - 40$$

$$P'(q) = -500\, q^{-2} - 0.5 \cdot 2q^{2-1} + 495 = \frac{-500}{q^2} - q + 495$$

$$P'(5) = \frac{-500}{5^2} - 5 + 495 = 470$$

The marginal profit when the demand is 5 units is $\underline{\$470}$

(C) Find the marginal profit when the demand is 500. Interpret you answer.

$$P'(q) = \frac{-500}{q^2} - q + 495$$

$$P'(500) = \frac{-500}{(500)^2} - 500 + 495 = -5.002$$

The marginal profit is $-\$5.002$ when the demand is 500.

Thus, the profit decreases by \$5.002 per item demanded when 500 items are demanded

Example. Suppose the profit in dollars of producing x items of a product is given by

$$P(x) = 2x - \frac{x^2}{400}. = 2x - \frac{1}{400}(x^2)$$

(A) What values of x makes marginal profit equal to 0?

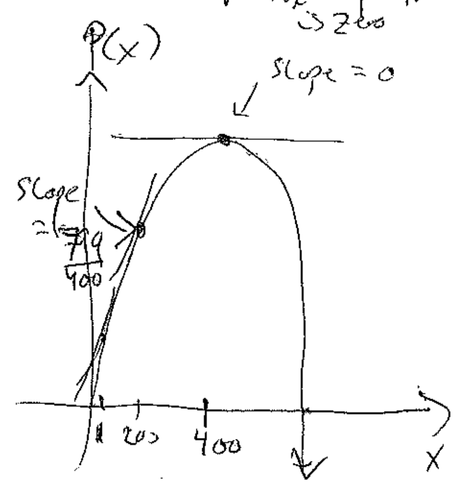
$$P'(x) = 2 - \frac{1}{400} 2x^{2-1} = 2 - \frac{1}{200} x$$

Put $P'(x) = 0$ and solve for x. $2 - \frac{1}{200} x = 0$

$400 - x = 0$

$x = 400$

When $x = 400$, the marginal profit is equal to 0.



(B) For what value of x does the profit increase by \$1 per item?

Put $P'(x) = 1$ and solve for x.

$2 - \frac{1}{200} x = 1$

$400 - x = 200$

$x = 200$

The marginal profit is \$1 when $x = 200$

(C) What is the equation of the tangent line to the graph of $P(x)$ at $x = 1$?

$y = mx + b$

$y = (P'(1))x + b$

$P'(x) = 2 - \frac{1}{200} x$

$P'(1) = 2 - \frac{1}{200} \cdot 1 = \frac{399}{200}$

$P(1) = 2 \cdot 1 - \frac{1^2}{400} = \frac{799}{400}$

$\frac{799}{400} = \frac{399}{200} \cdot 1 + b$

$b = \frac{799}{400} - \frac{399}{200} = \frac{1}{400}$

Point $(1, \frac{799}{400})$

$y = \frac{399}{200} \cdot x + \frac{1}{400}$