3.4: Definition of the Derivative

Recall the following:

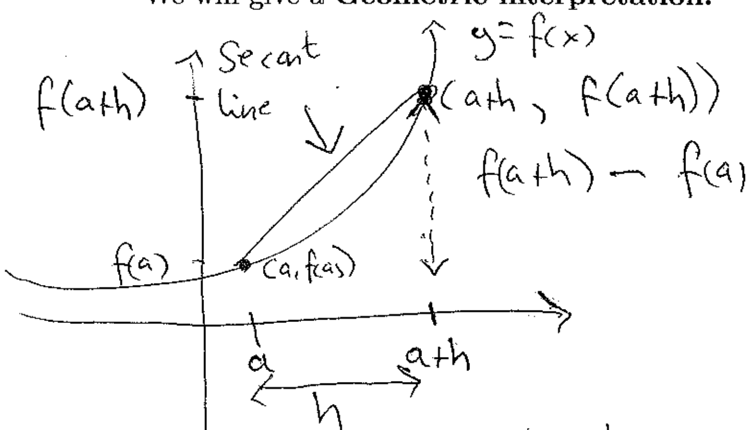
Instantaneous rate of change:

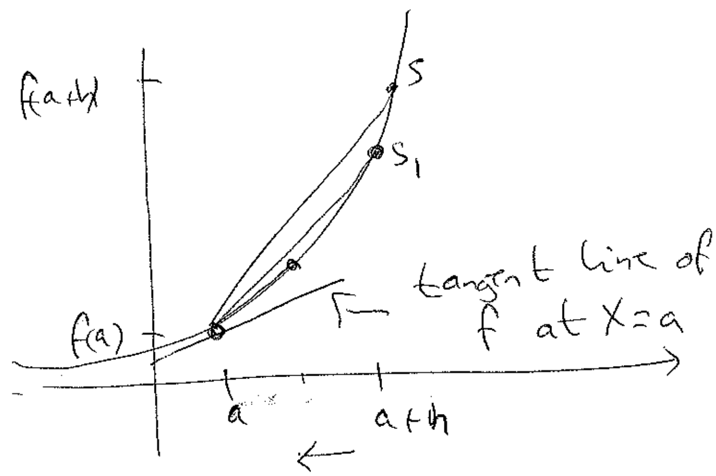
The **instantaneous rate of change** of $f(x)$ with respect to x when $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

We will give a **Geometric interpretation:**





The Secant line is the line joining the points $(a, f(a))$ and $(a+h, f(a+h))$

Slope of Secant line: $\frac{\Delta y}{\Delta X} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$

The slope of the Secant line corresponds to the average rate of change of y with respect to X over the interval $[a, a+h]$

If the slopes of the secant lines approach a limit as h approaches 0, then this limit is defined to be the slope of the tangent line of f at the point $X = a$.

Slope of the tangent line:

The **tangent line** of the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through this point having slope

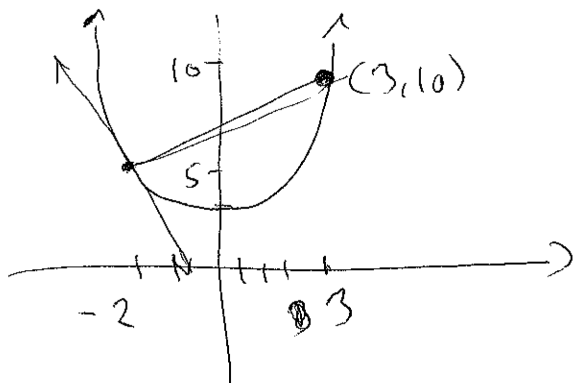
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

- The slope of the tangent line of the graph of $y = f(x)$ at the point $(a, f(a))$ is the same as the instantaneous rate of change of f at $x = a$.
- The slope of the tangent line at a point is also called the **slope of the curve** at the point.

Example. Let $f(x) = x^2 + 1$

A. Find the slope and the equation of the secant line through the points where $x = -2$ and $x = 3$.



$f(-2) = (-2)^2 + 1 = 5$

$f(3) = 10$

The slope of the secant line, $\frac{f(3) - f(-2)}{3-(-2)} = \frac{10-5}{5} = 1$

Equation:

$y = mx + b$ $\quad m = 1$

$(y - 10) = 1 \cdot (x - 3)$

$y = x - 3 + 10$

$y = x + 7$

$y = x^2 + 1$

B. Find the slope and equation of the tangent line at $x = -2$.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad a = -2$$

$$\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h \to 0} \frac{(-2+h)^2 + 1 - ((-2)^2 + 1)}{h}$$

$$= \lim_{h \to 0} \frac{4 - 4h + h^2 + 1 - 4 - 1}{h}$$

$$= \lim_{h \to 0} \frac{-4h + h^2}{h} = \lim_{h \to 0} -4 + h$$

$$= -4 + 0 = -4$$ is the slope of the tangent line at $x = -2$.

$m = -4$

$y = mx + b$

$f(-2) = (-2)^2 + 1 = 5$

$(-2, 5)$ is on the graph

$m = -4$

$y - 5 = -4(x - (-2))$

$y = -4(x + 2) + 5$

$y = -4x - 8 + 5$

$y = -4x - 3$

The Derivative:

We denote

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

The Derivative:

The **derivative** of the function f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

The function $f'(x)$ is called the derivative of f with respect to x.

The notation $f'(x)$ is read 'f- prime of x'.

Notice that $f'(x)$ is a **function** of x as x varies.

$f'(a)$ is the slope of the tangent line at $x = a$ which is a number. $f'(a)$ is the value of $f'(x)$ evaluated at $x = a$.

If $f'(x)$ exists, we say that f is **differentiable** at x. The process that gives f' is called **differentiation.**

Interpretations of the derivative:

1. The function $f'(x)$ represents the **instantaneous rate of change** of $y = f(x)$ with respect to x. From now on we will say **rate of change** to mean **instantaneous rate of change**.

2. The function $f'(x)$ represents the **slope** of the graph of $f(x)$ at any point x. If we evaluate the derivative at $x = a$, to get $f'(a)$, then $f'(a)$ represents the slope of the curve or the slope of the tangent line at that point.

The difference quotient,

$$\frac{f(x+h)-f(x)}{h} \qquad \text{represents}$$

:

- Slope of the secant line
- Average rate of change
- Average rate of change in cost, revenue, or profit
- Average velocity

The derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{represents}$$

:

- Slope of the tangent line
- Instantaneous rate of change
- Marginal cost, revenue, or profit
- Instantaneous velocity

Let $b = x + h$ so $h = b - x$. Then we have the following alternate form of the derivative:

h -> 0 , then
b -> x

The Derivative:

The **derivative** of the function f at x can be written as

$$f'(x) = \lim_{b \to x} \frac{f(b)-f(x)}{b-x}$$

provided this limit exists.

Example. Let $f(x) = \frac{2}{x}$

A. Find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 1. $f(x+h) - f(x) = \frac{2}{x+h} - \frac{2}{x} = \frac{2x - 2(x+h)}{(x+h)x}$

$= \frac{2x - 2x - 2h}{(x+h)x}$

$= \frac{-2h}{(x+h)\cdot x}$

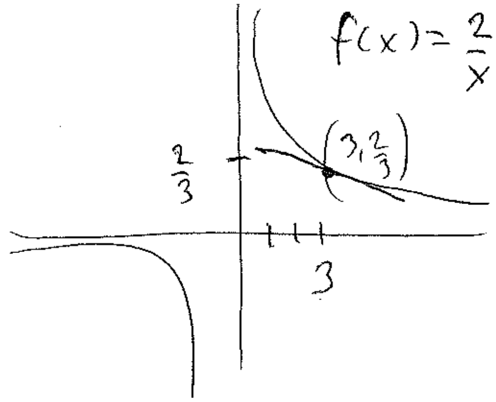
Step 2 $\frac{f(x+h) - f(x)}{h}$ Difference quotient

$= \frac{\frac{-2h}{(x+h)\cdot x}}{h} = \frac{-2h}{(x+h)\cdot x \cdot h} = \frac{-2}{(x+h)\cdot x}$

Step 3 $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-2}{(x+h)x} = \frac{-2}{(x+0)\cdot x} = \frac{-2}{x^2}$

$f'(x) = \frac{-2}{x^2}$

$f(x) = \frac{2}{x}$



B. Find $f'(3)$

$f'(x) = -\frac{2}{x^2}$

$f'(3) = -\frac{2}{3^2} = -\frac{2}{9}$

C. Find the equation of the tangent line at $x = 3$.

$m = -\frac{2}{9}$ $\quad f(3) = \frac{2}{3}$

Equation with slope $m = -\frac{2}{9}$ and point $(3, \frac{2}{3})$

$y - \frac{2}{3} = -\frac{2}{9}(x-3)$

$y = -\frac{2}{9}x + \frac{2}{3} + \frac{2}{3}$

$y = -\frac{2}{9}x + \frac{4}{3}$

Using the point-slope form we obtain

Equation of the Tangent line:

The tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ is given by the equation

$$y - f(x_1) = f'(x_1)(x - x_1)$$

provided $f'(x)$ exists.

$f'(x_1) = m =$ slope at x_1

Example. The profit, P, in (thousands of dollars) from the expenditure of x thousand dollars on advertising is given by

$$P(x) = 1000 + 90x - x^2$$

(A) Find the marginal profit at the following expenditures: Decide in each case, whether the firm should increase the expenditure:

1. \$4000 One way: $\lim_{h \to 0} \frac{P(4+h) - P(4)}{h}$ Marginal profit at $X = 4$

Second way:
First find $P'(x)$
Then plug in $X=4$ to find $P'(4)$.

$$P'(x) = \lim_{h \to 0} \frac{P(x+h) - P(x)}{h}$$

$$= \lim_{h \to 0} \frac{1000 + 90(x+h) - (x+h)^2 - (1000 + 90x - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{90h - (x^2 + 2xh + h^2) + x^2}{h} = \lim_{h \to 0} \frac{90h - 2xh - h^2}{h}$$

$$= \lim_{h \to 0} 90 - 2x - h = 90 - 2X$$

$$P'(x) = 90 - 2X$$

$$P'(4) = 90 - 2 \cdot 4 = 82.$$

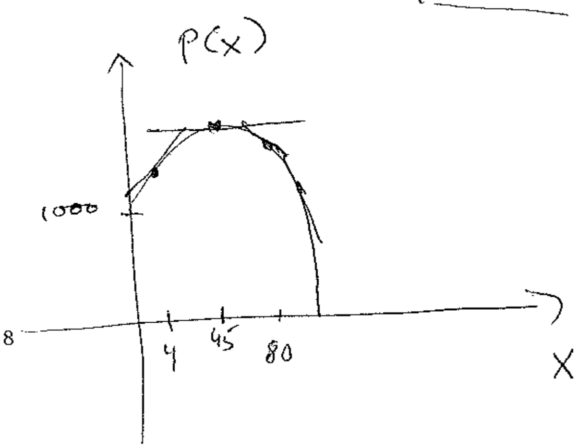
The marginal profit is \$ 82000

Since $P'(4) > 0$, increase expenditure

2. \$80000

$$P'(80) = 90 - 2 \cdot 80 = -70 < 0$$

Marginal profit is $-\$70{,}000$



do not increase expenditure since $P'(80) < 0$

Existence of the derivative

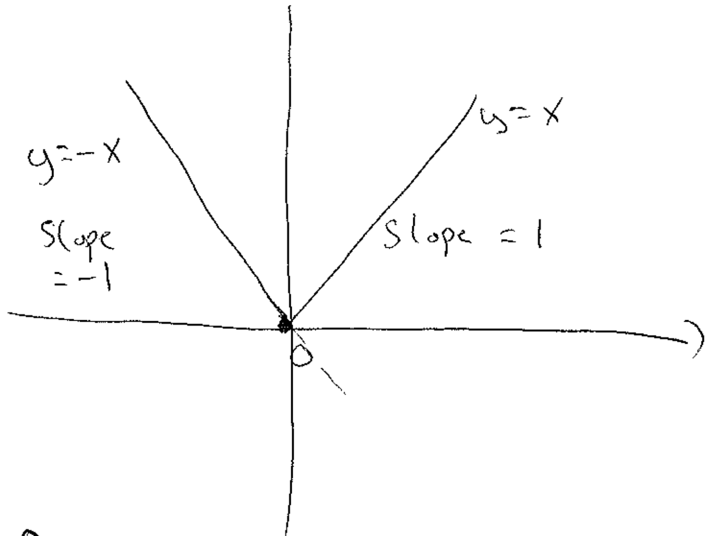
The derivative $f'(x)$ of a function f does not always exists. Here are some examples:

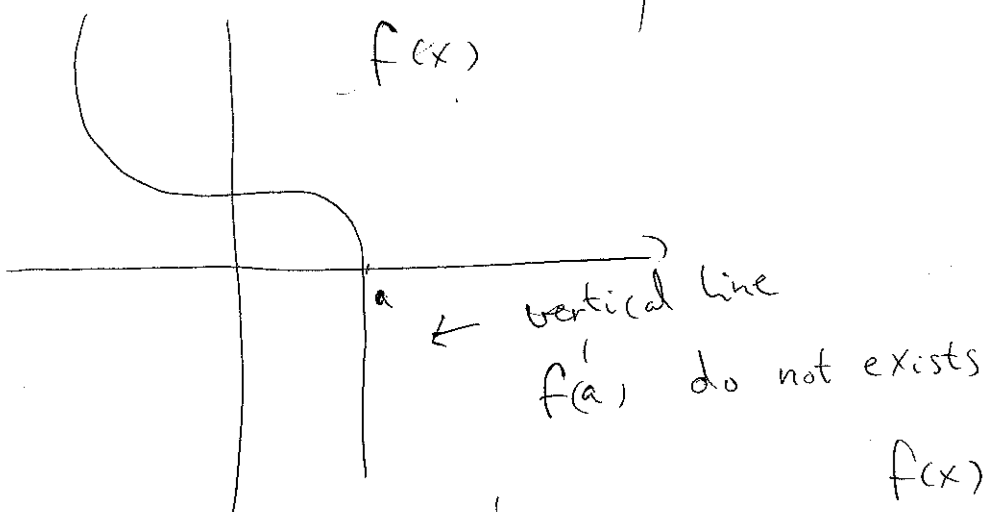
Example. Let

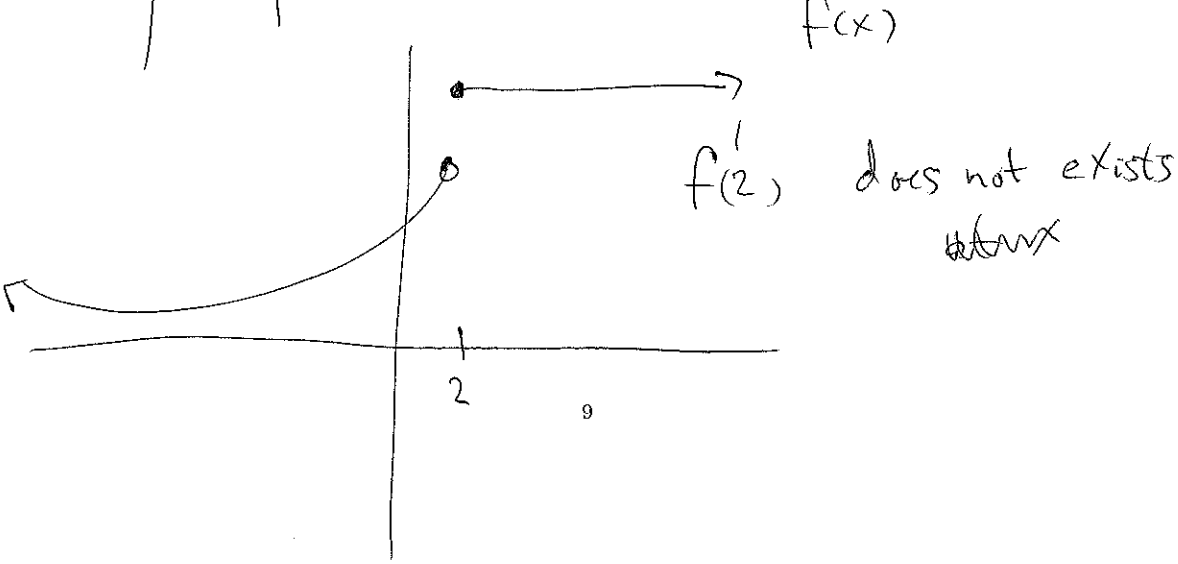
$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Does $f'(0)$ exists?

no







The derivative exists when a function f satisfies **all** of the following conditions at a point:

- f is continuous
- f is smooth
- f does not have a vertical tangent line.

The derivative does not exists when **any** of the following conditions are true for a function at a point:

- f is discontinuous
- f has a sharp corner
- f has a vertical tangent line.