3.4: Definition of the Derivative

Recall the following:

Instantaneous rate of change:

The **instantaneous rate of change** of $f(x)$ with respect to x when $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

We will give a **Geometric interpretation:**

Slope of the tangent line:

The **tangent line** of the graph of $y = f(x)$ at the point $(a, f(a))$ is the line through this point having slope

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

- The slope of the tangent line of the graph of $y = f(x)$ at the point $(a, f(a))$ is the same as the instantaneous rate of change of f at $x = a$.
- The slope of the tangent line at a point is also called the **slope of the curve** at the point.

Example. Let $f(x) = x^2 + 1$

A. Find the slope and the equation of the secant line through the points where $x = -2$ and $x = 3$.

B. Find the slope and equation of the tangent line at $x = -2$.

The Derivative:

We denote

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

The Derivative:

The **derivative** of the function f at x is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

The function $f'(x)$ is called the derivative of f with respect to x.

The notation $f'(x)$ is read 'f- prime of x'.

Notice that $f'(x)$ is a **function** of x as x varies.

$f'(a)$ is the slope of the tangent line at $x = a$ which is a number. $f'(a)$ is the value of $f'(x)$ evaluated at $x = a$.

If $f'(x)$ exists, we say that f is **differentiable** at x. The process that gives f' is called **differentiation**.

Interpretations of the derivative:

1. The function $f'(x)$ represents the **instantaneous rate of change** of $y = f(x)$ with respect to x. From now on we will say **rate of change** to mean **instantaneous rate of change**.

2. The function $f'(x)$ represents the **slope** of the graph of $f(x)$ at any point x. If we evaluate the derivative at $x = a$, to get $f'(a)$, then $f'(a)$ represents the slope of the curve or the slope of the tangent line at that point.

The difference quotient,

$$\frac{f(x+h)-f(x)}{h} \qquad \text{represents}$$

:

- Slope of the secant line
- Average rate of change
- Average rate of change in cost, revenue, or profit
- Average velocity

The derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{represents}$$

:

- Slope of the tangent line
- Instantaneous rate of change
- Marginal cost, revenue, or profit
- Instantaneous velocity

Let $b = x + h$ so $h = b - x$. Then we have the following alternate form of the derivative:

The Derivative:

The **derivative** of the function f at x can be written as

$$f'(x) = \lim_{b\to x} \frac{f(b)-f(x)}{b-x}$$

provided this limit exists.

Example. Let $f(x) = \frac{2}{x}$

A. Find $f'(x)$

B. Find $f'(3)$

C. Find the equation of the tangent line at $x = 3$.

Using the point-slope form we obtain

Equation of the Tangent line:

The tangent line to the graph of $y = f(x)$ at the point $(x_1, f(x_1))$ is given by the equation

$$y - f(x_1) = f'(x_1)(x - x_1)$$

provided $f'(x)$ exists.

Example. The profit, P, in (thousands of dollars) from the expenditure of x thousand dollars on advertising is given by

$$P(x) = 1000 + 90x - x^2$$

(A) Find the marginal profit at the following expenditures: Decide in each case, whether the firm should increase the expenditure:

1. \$4000

2. \$80000

Existence of the derivative

The derivative $f'(x)$ of a function f does not always exists. Here are some examples:

Example. Let

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Does $f'(0)$ exists?

The derivative exists when a function f satisfies **all** of the following conditions at a point:

- f is continuous
- f is smooth
- f does not have a vertical tangent line.

The derivative does not exists when **any** of the following conditions are true for a function at a point:

- f is discontinuous
- f has a sharp corner
- f has a vertical tangent line.