3.3: Rates of change

Example. Suppose a car starts traveling along a straight road. Assume the distance traveled by the car is given by the function,

$$S(t) = 3t^2 \quad \text{for} \quad 0 \le t \le 4,$$

where t is time in seconds and $S(t)$ is the distance in feet. We record the distance traveled every second from 0 to 4 second. We obtained the following:

t (sec.)	0	1	2	3	4
$S(t)$ (feet)	0	3	12	27	48

During the first second ($0 \le t \le 1$), the car has traveled 3 ft.

During the next second ($1 \le t \le 2$), the car has traveled 9 ft.

During the time interval $2 \le t \le 3$ seconds, the car has traveled 15 ft. $27-12=15$

Recall the formula:

$$\textbf{average speed} = \frac{\textbf{distance}}{\textbf{time}}$$

During the first second ($0 \le t \le 1$), the **average speed** is 3 ft/sec. $\frac{3}{1}=3$

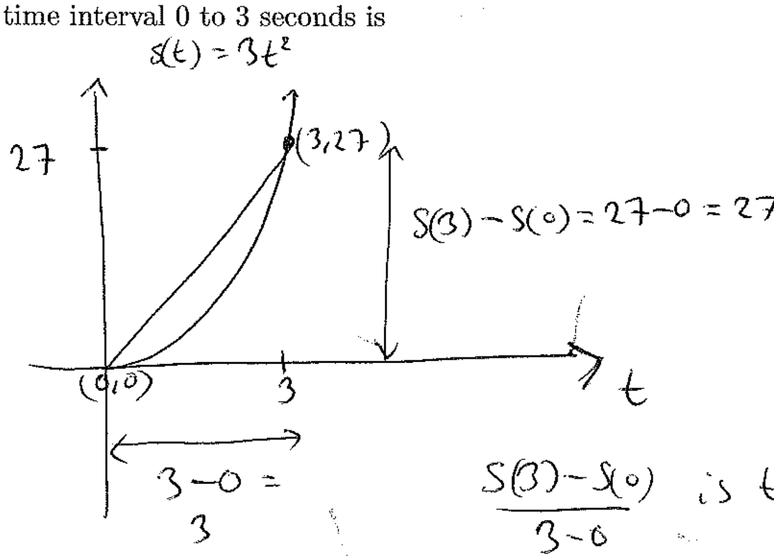
During the next second ($1 \le t \le 2$), the **average speed** is 9 ft/sec. $\frac{9}{1}=9$

During the time interval $2 \le t \le 3$ seconds, the **average speed** is 15 ft/sec. $\frac{15}{1}=15$

The **average speed** over the time interval 0 to 3 seconds is

$$\frac{S(3)-S(0)}{3-0} = \frac{3\cdot 3^2 - 0}{3} = \frac{27}{3} = 9$$

The average speed is 9 ft/sec over the interval [0,3].



$\frac{S(3)-S(0)}{3-0}$ is the slope of the line joining the points (0,0) and (3,27).

$$\text{average speed over a time interval} = \frac{\text{change in distance}}{\text{change in time}} = \frac{\text{change in y}}{\text{change in x}} = \text{slope}$$

The **speed** is the **average rate of change** of distance, s, with respect to time, t.

Speed it the magnitude of the velocity and is always positive.

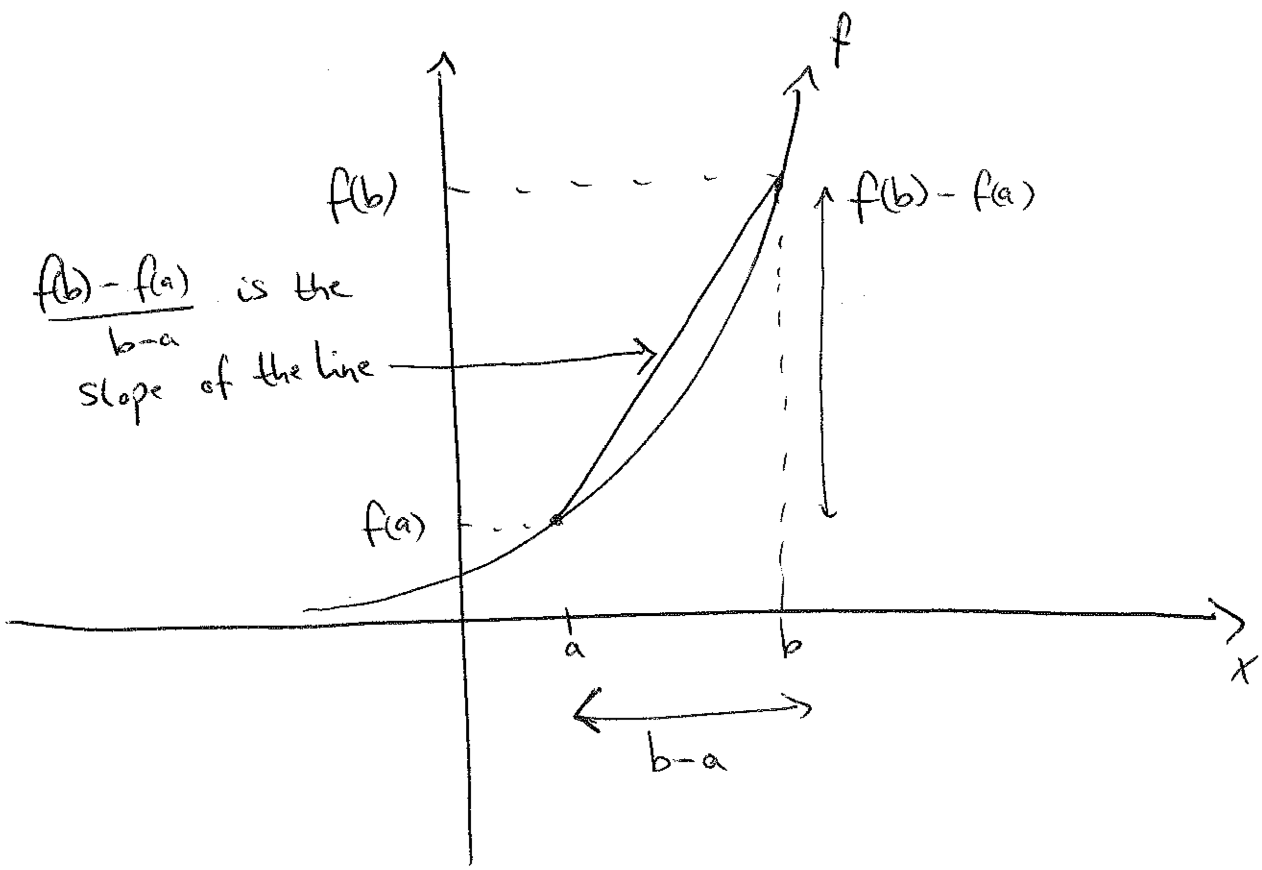
We can calculate the average rate of change for any function:

Average rate of change:

The **average rate of change** of $f(x)$ with respect to x for a function f over the interval, $a \leq x \leq b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

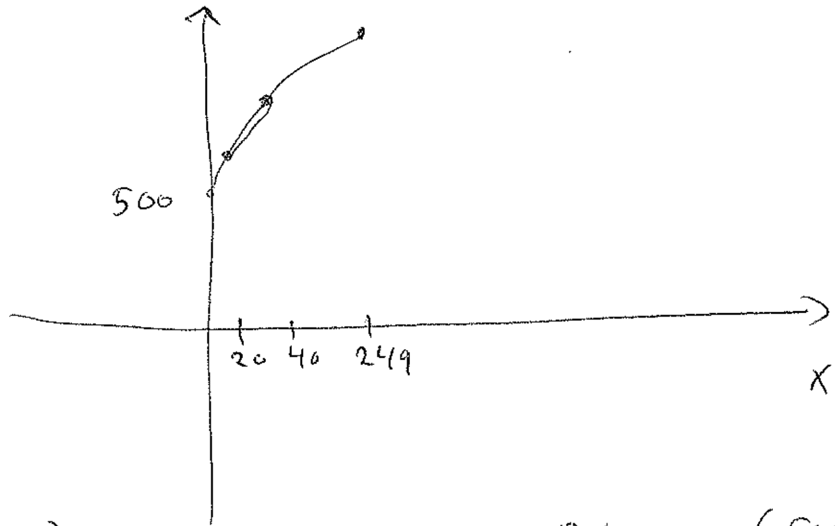
The average rate of change of f over the interval $[a, b]$ is the slope of the line segment joining the points $(a, f(a))$ and $(b, f(b))$.



Example. Suppose the cost, C, in dollars of producing x electric guitars is given by

$$C(x) = 500 + 249x - 0.5x^2 \qquad 0 \le x \le 249$$

What is the average rate of change in cost as the production changes from 20 to 40 guitars?



$$\frac{C(40) - C(20)}{40 - 20} = \frac{500 + 249 \cdot 40 - 0.5 \cdot 40 \cdot 40 - (500 + 249 \cdot 20 - 0.5 \cdot 20^2)}{20}$$

$$= 219$$

The average rate of change in cost as the production changes from 20 to 40 guitars is \$ 219 per guitar

Thus, on average the cost increases at the rate of \$ 219 per guitar when production increases from 20 to 40 guitars.

Example. Suppose a car starts traveling along a straight road. Assume the distance traveled by the car is given by the function,

$$S(t) = 3t^2 \quad \text{for} \quad 0 \le t \le 4,$$

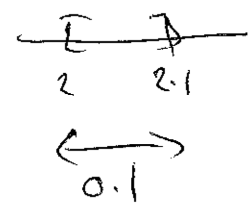
where t is time in seconds and $S(t)$ is the distance in feet.

What is the **exact speed** of the car at the instant, $t = 2$ seconds?

We will take smaller and smaller intervals near t = 2 and calculate the average speed over these intervals:

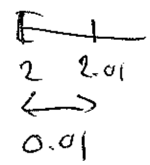
Over the time interval, $t = 2$ to $t = 2.1$, the average speed is 12.3 ft/sec

$$\frac{S(2.1) - S(2)}{2.1 - 2} = \frac{3 \cdot (2.1)^2 - 3 \cdot 2^2}{0.1} = 12.3$$



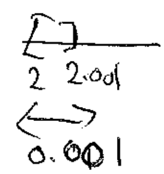
Over the time interval, $t = 2$ to $t = 2.01$, the average speed is 12.03 ft/sec

$$\frac{S(2.01) - S(2)}{2.01 - 2} = 12.03$$

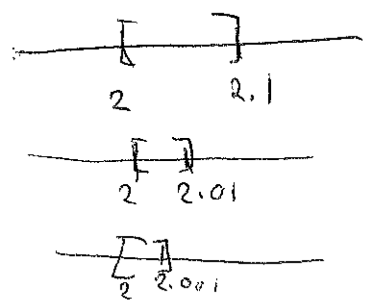


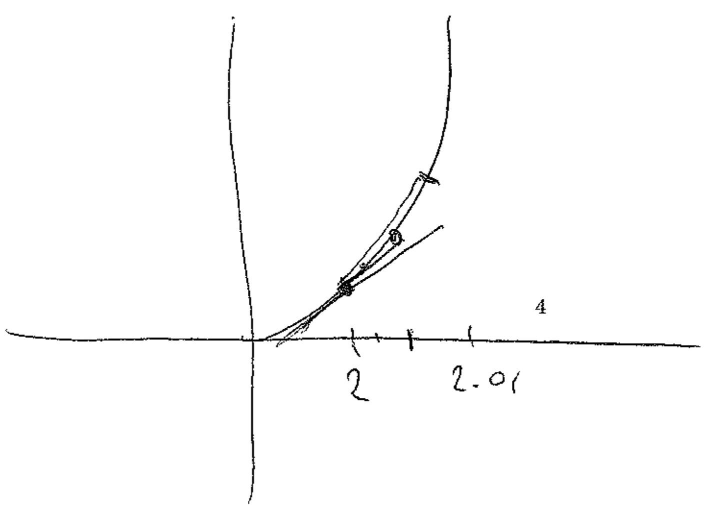
Over the time interval, $t = 2$ to $t = 2.001$, the average speed is 12.003 ft/sec

$$\frac{S(2.001) - S(2)}{2.001 - 2} = 12.003$$



The **exact speed** at $t = 2$ seconds is 12 ft/sec.





By taking smaller and smaller intervals near $t = 2$, the average speed over these intervals should get closer and closer to the exact speed at $t = 2$ seconds.

Thus, the exact speed at $t = 2$ second is the limit of the average speeds over shorter and shorter time intervals near $t = 2$.

We compute the average speed from $t = 2$ to $t = 2 + h$, where h is a small, nonzero number that represents a small change in time. (In the previous problem h was respectively, 0.1, 0.01, and 0.001.

$S(t) = 3t^2$

The average speed from $t = 2$ to $t = 2 + h$ is given by $12 + 3h$

$$\frac{S(2+h) - S(2)}{(2+h) - 2} = \frac{3 \cdot (2+h)^2 - 3 \cdot 2^2}{h}$$

$$= \frac{3 \cdot (4 + 4h + h^2) - 12}{h}$$

$$= \frac{12 + 12h + 3h^2 - 12}{h}$$

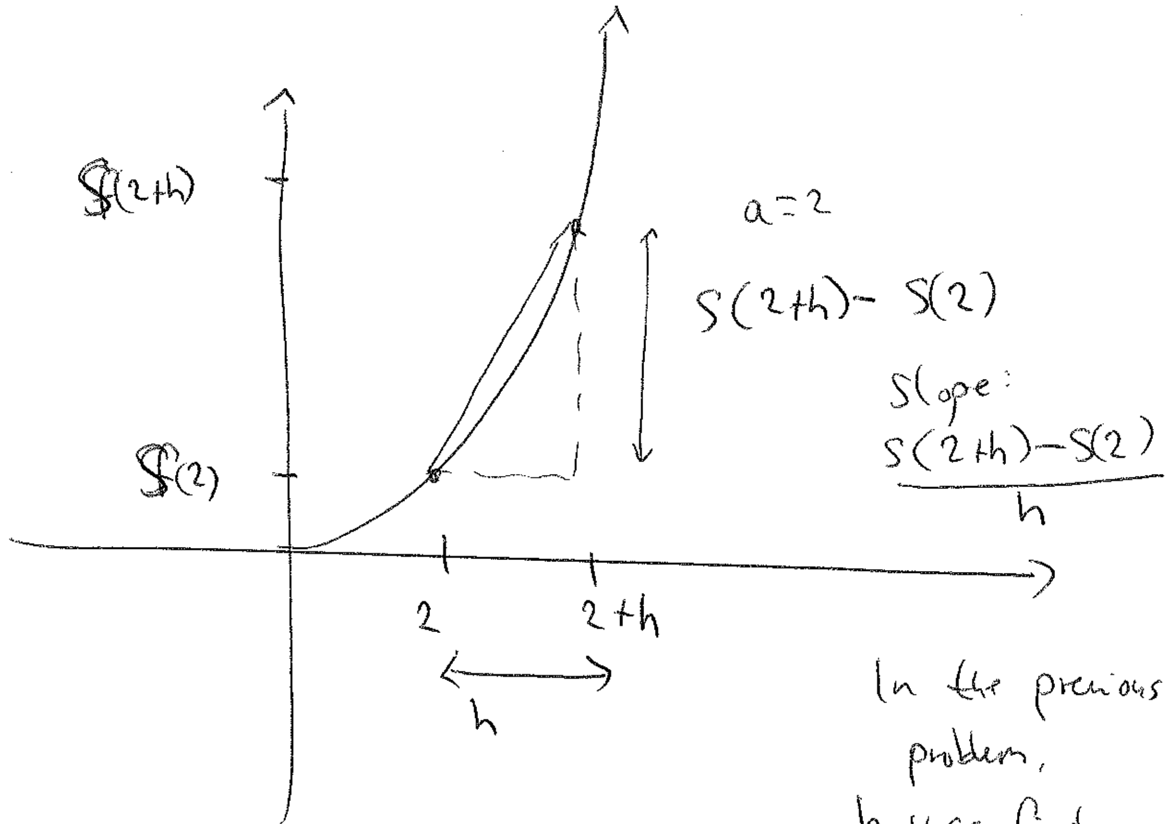
$$= \frac{12h + 3h^2}{h} = 12 + 3h$$

Taking the intervals from 2 to $2 + h$ to be shorter and shorter is equivalent to saying that h gets closer and closer to 0.

Thus, the **exact speed** at $t = 2$ seconds is

$$\lim_{h \to 0} \frac{S(2+h) - S(2)}{(2+h) - 2} = \lim_{h \to 0} 12 + 3h = 12 + 3 \cdot 0$$

$$= 12$$



In the previous problem, h was first 0.1, then it was 0.01, then 0.001

Now let f be any function. Let a be a specific x-value. Let h be a small, positive number which represents the distance between the two values of x, which are a and $a+h$. Then the average rate of change of f as x changes from a to $a+h$ is

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}.$$

← Slope of the line through the points $(a, f(a))$ and $(a+h, f(a+h))$

The last expression is called the **difference quotient**.

Instantaneous rate of change:

The **instantaneous rate of change** of $f(x)$ with respect to x when $x = a$ is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

provided this limit exists.

The **exact rate of change** of f at $x = a$ is called the **instantaneous rate of change** of f at $x = a$.

- Instantaneous rate of change can be positive, zero or negative.
- Velocity is the instantaneous rate of change of a function that gives the position as a function of time.
- Velocity is positive in one direction and negative in the opposite direction.

Now let $a + h = b$ so $h = b - a$, then we have the following alternate approach to find the instantaneous rate of change:

As $b \to a$, then $h \to 0$

Instantaneous rate of change:

The **instantaneous rate of change** of $f(x)$ with respect to x when $x = a$ is

$$\lim_{b\to a} \frac{f(b)-f(a)}{b-a}$$

provided this limit exists.

The **marginal cost** is the instantaneous rate of change of the cost function with respect to the production at a given production level.

The **marginal revenue** is the instantaneous rate of change of the revenue function with respect to the production at a given production level.

The **marginal profit** is the instantaneous rate of change of the profit function with respect to the production at a given production level.

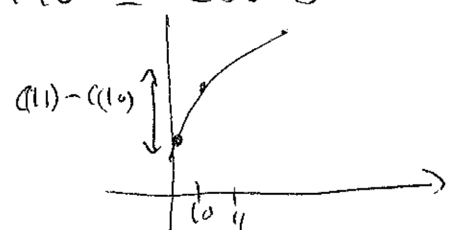
Example. Suppose the cost, C, in dollars of producing x electric guitars is given by

$$C(x) = 500 + 249x - 0.5x^2 \quad 0 \leq x \leq 249.$$

(A) Find the additional cost when production is increased from 10 to 11 electric guitars.

$C(11) - C(10) = 3178.5 - 2940 = 238.5$

Thus, the additional cost when production is increased from 10 to 11 guitars is \$238.5



(B) Calculate the marginal cost to produce 10 electric guitars. That is find the instantaneous rate of change of cost with respect to the number of electric guitars produced when 10 electric guitars are produced. Interpret the result.

$$\lim_{h \to 0} \frac{C(10+h) - C(10)}{h} =$$

$$\lim_{h \to 0} \frac{500 + 249 \cdot (10+h) - 0.5(10+h)^2 - (500 + 249 \cdot 10 - 0.5 \cdot 10^2)}{h}$$

$$= \lim_{h \to 0} \frac{2490 + 249h - 0.5(100 + 20h + h^2) - 2490 + 50}{h}$$

$$= \lim_{h \to 0} \frac{249h - 10h - 0.5h^2}{h} = \lim_{h \to 0} 249 - 10 - 0.5h$$

$$= 239 - 0.5 \cdot 0 = 239$$

The marginal cost is \$239 when 10 guitars are produced

When 10 guitars are produced, the cost is increasing at the rate of \$239 per guitar.

When 10 guitars are produced it cost \$239 to produce an additional guitar.