3.2: Continuity

Continuity at $x = c$:

A function f is **continuous** at $x = c$ if the following three conditions are satisfied:

- 1. $f(c)$ is defined.
- 2. $\lim\limits_{x \to c} f(x)$ exists, and
- 3. $\lim\limits_{x \to c} f(x) = f(c)$.

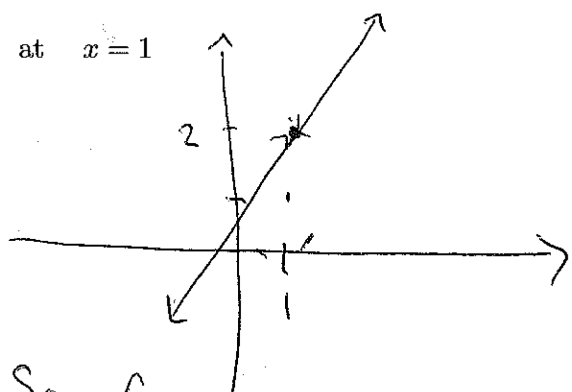
If f is not continuous at c, it is **discontinuous** there.

We will use this 3- step test to check if a function is continuous:

Example. For the following functions, draw the graph. Then determine if the functions are continuous at the indicated x-value:

1.

$$f(x) = x + 1 \quad \text{at} \quad x = 1$$



Step 1 $f(1) = 1 + 1 = 2$ So f is defined at $X = 1$

Step 2 $\lim\limits_{X \to 1} f(x) = \lim\limits_{X \to 1} (X + 1) = 1 + 1 = 2$

So $\lim\limits_{X \to 1} f(x)$ exists

Step 3 From step 2

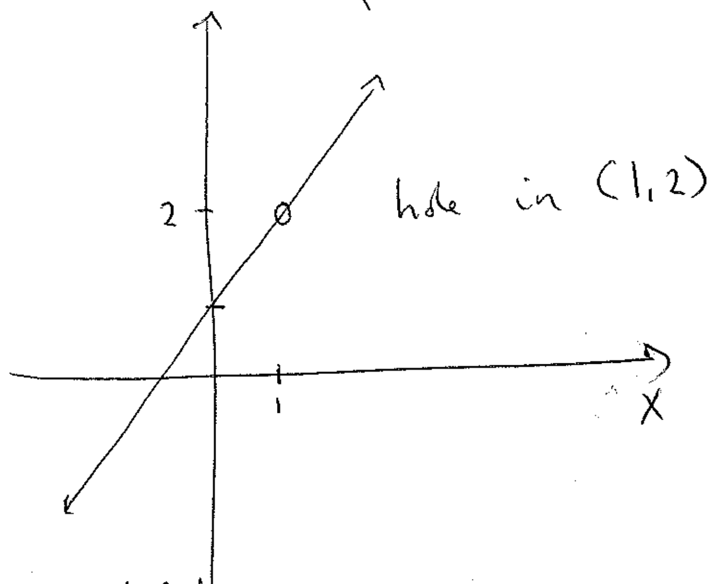
$$\lim_{X \to 1} f(x) \overset{\downarrow}{=} 2 = f(1)$$

So $f(x)$ is continuous at $X = 1$

2.

$$g(x) = \frac{x^2 - 1}{x - 1} \quad \text{at} \quad x = 1$$

$$= \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1 \qquad \text{for } x \neq 1$$



Step 1 $\quad g(1)$ is not defined

So $\; g$ is not continuous at $x=1$

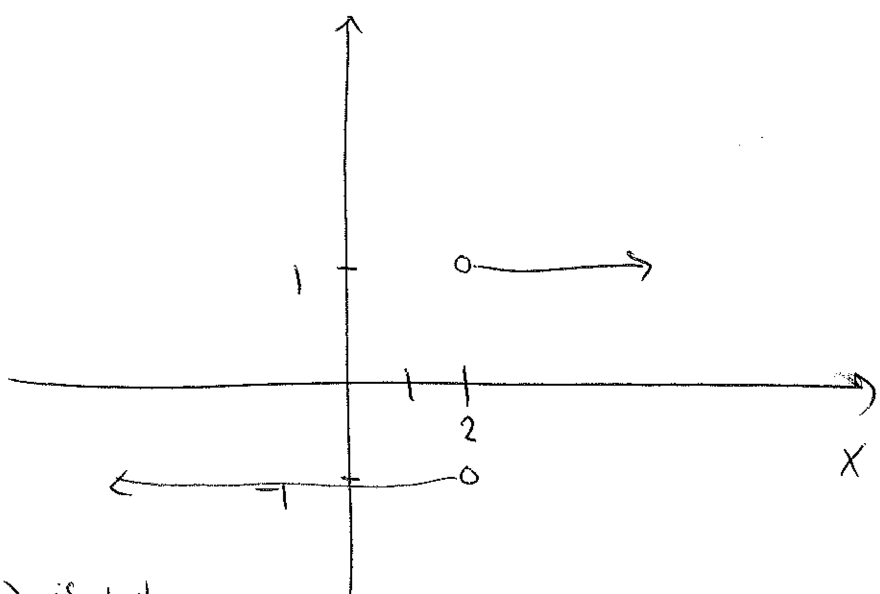
3.

$$h(x) = \frac{|x-2|}{x-2} \quad \text{at} \quad x = 2$$

$$|X-2| = \begin{cases} X-2, & X \geq 2 \\ -(X-2), & X < 2 \end{cases}$$

So $h(x) = \dfrac{|X-2|}{X-2} = \begin{cases} \dfrac{X-2}{X-2}, & X > 2 \\ \dfrac{-(X-2)}{X-2}, & X < 2 \end{cases}$

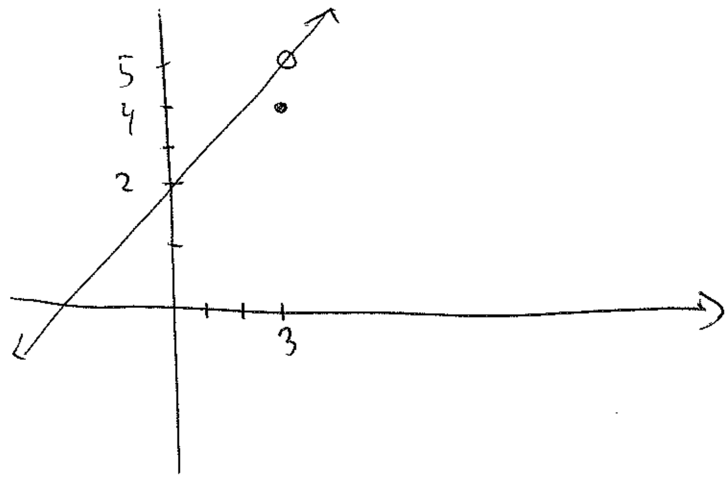
$$= \begin{cases} 1, & X > 2 \\ -1, & X < 2 \end{cases}$$



Step 1. $h(x)$ is not defined at $X=2$, So $h(x)$ is discontinuous at $X=2$.

4.

$$k(x) = \begin{cases} x+2 & \text{if } x \neq 3; \\ 4 & \text{if } x = 3. \end{cases} \quad \text{at} \quad x = 3$$



Step 1 $\quad h(3) = 4$ So $k(x)$ is defined at $X = 3$

Step 2. $\quad \lim_{X \to 3^-} k(x) = \lim_{X \to 3^-} X+2 = 3+2=5$

$\lim_{X \to 3^+} k(x) = \lim_{X \to 3^+} X+2 = 3+2=5$

Hence $\lim_{X \to 3} k(x) = 5$ So the $\lim_{X \to 3} k(x)$ exists.

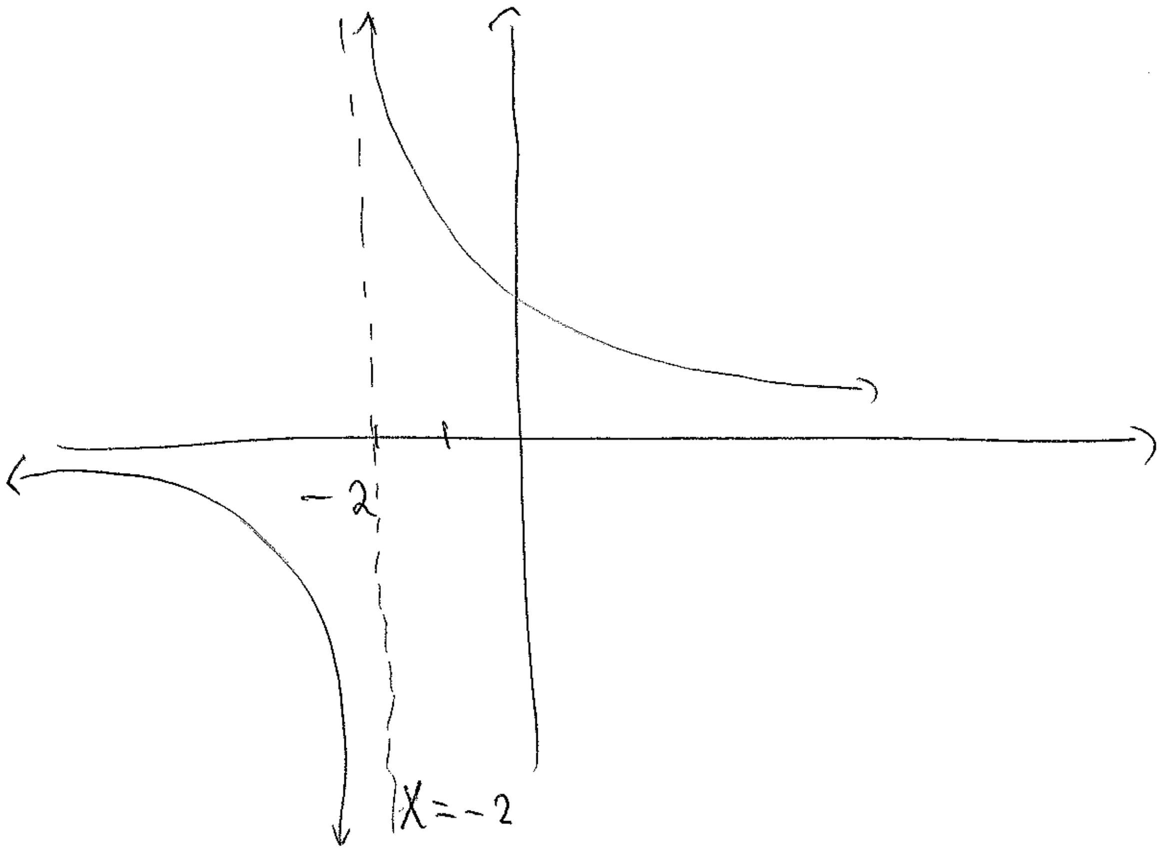
Step 3. We see that

$$4 = k(3) \neq \lim_{X \to 3} k(x) = 5$$

So $k(x)$ is discontinuous at $X = 3$.

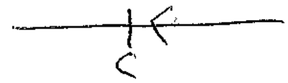
5.

$$l(x) = \frac{1}{x+2} \quad \text{at} \quad x = -2$$



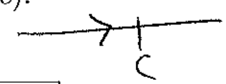
Step 1 $l(-2)$ is not defined
$X=-2$ is a vertical asymptote
l is dis continuous at $X=-2$.

Definition. A function is **continuous** on an open interval if it is continuous at every x-value in the interval.



Definition. A function is **continuous from the right** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$.

Definition. A function is **continuous from the left** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$.

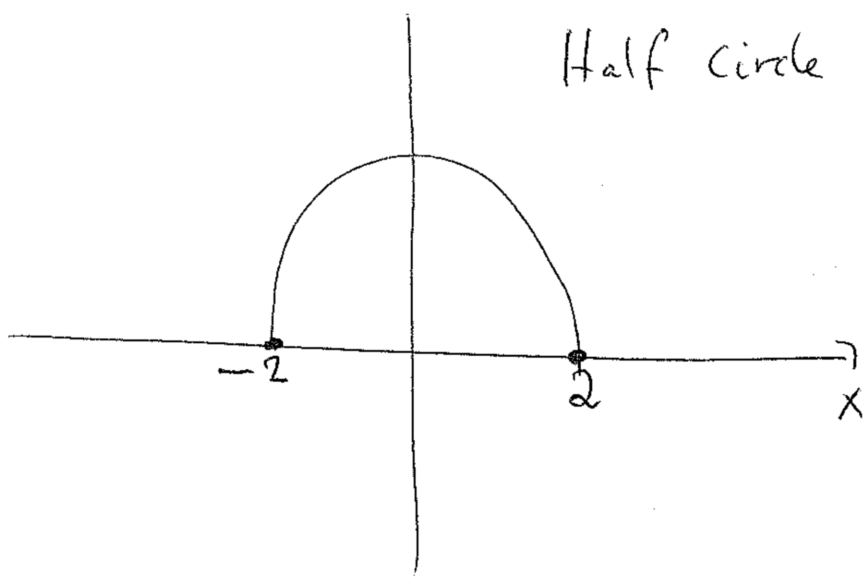


Continuity on a Closed interval:

A function f is **continuous on a closed interval** $[a, b]$ if:

- 1. it is continuous on the open interval (a, b).
- 2. it is continuous from the right at $x = a$, and
- 3. it is continuous from the left at $x = b$.

Example. The function $f(x) = \sqrt{4 - x^2}$ is continuous on the closed interval $[-2, 2]$.

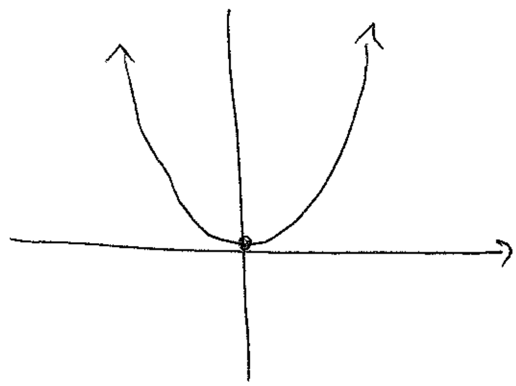


Here are the functions we have learned so far listed with the intervals in which the function is continuous:

Polynomial function, $y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers:

Continuous at: For all x

EX $y = x^2$



Rational function, $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with $q(x) \neq 0$:

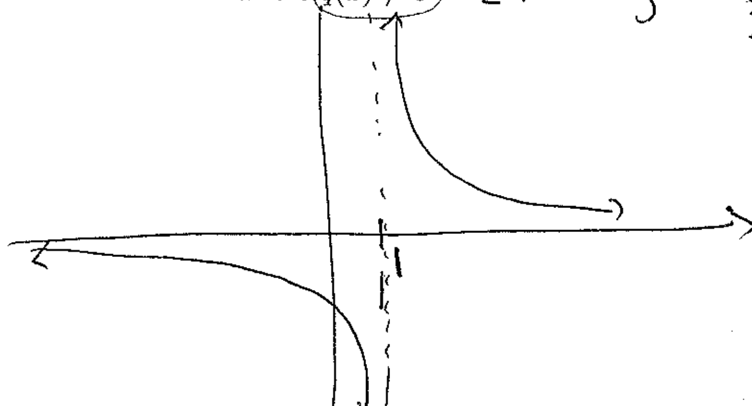
Continuous at: For all x where $q(x) \neq 0$

EX $y = \dfrac{1}{X-1}$

$X - 1 = 0$

$X = 1$

So discontinuous at $X = 1$



Root function, $y = \sqrt{ax+b}$, where a and b are real numbers with $a \neq 0$ and $ax + b \geq 0$:

Continuous at: For all x where $ax + b \geq 0$

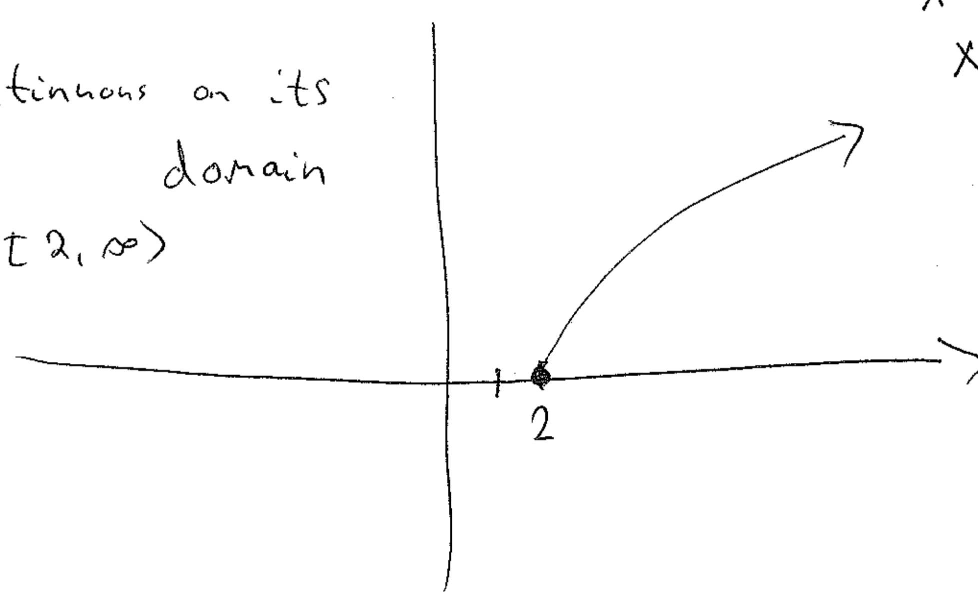
EX $y = \sqrt{X-2}$

$X - 2 \geq 0$

$X \geq 2$

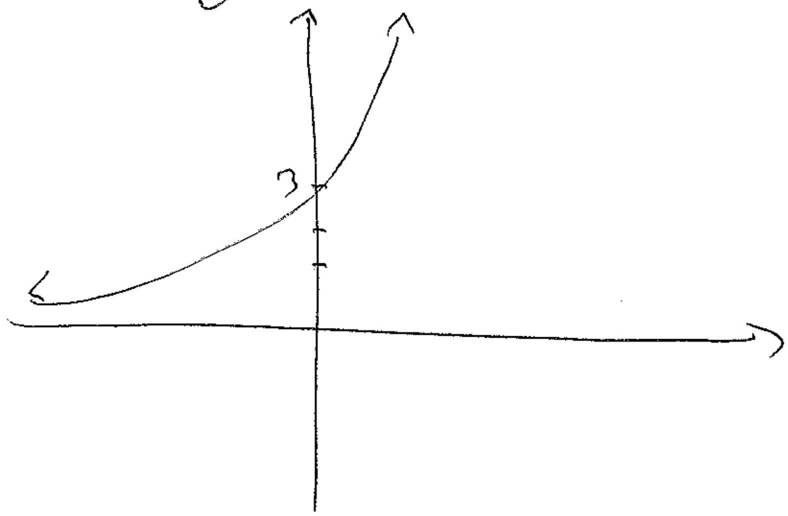
Continuous on its domain

$[2, \infty)$



Exponential function, $y = P_0 a^x$, where $a > 0$ and P_0 is the value of y at $x = 0$:

Continuous at: For all x $\quad y = 3 \cdot 2^x$

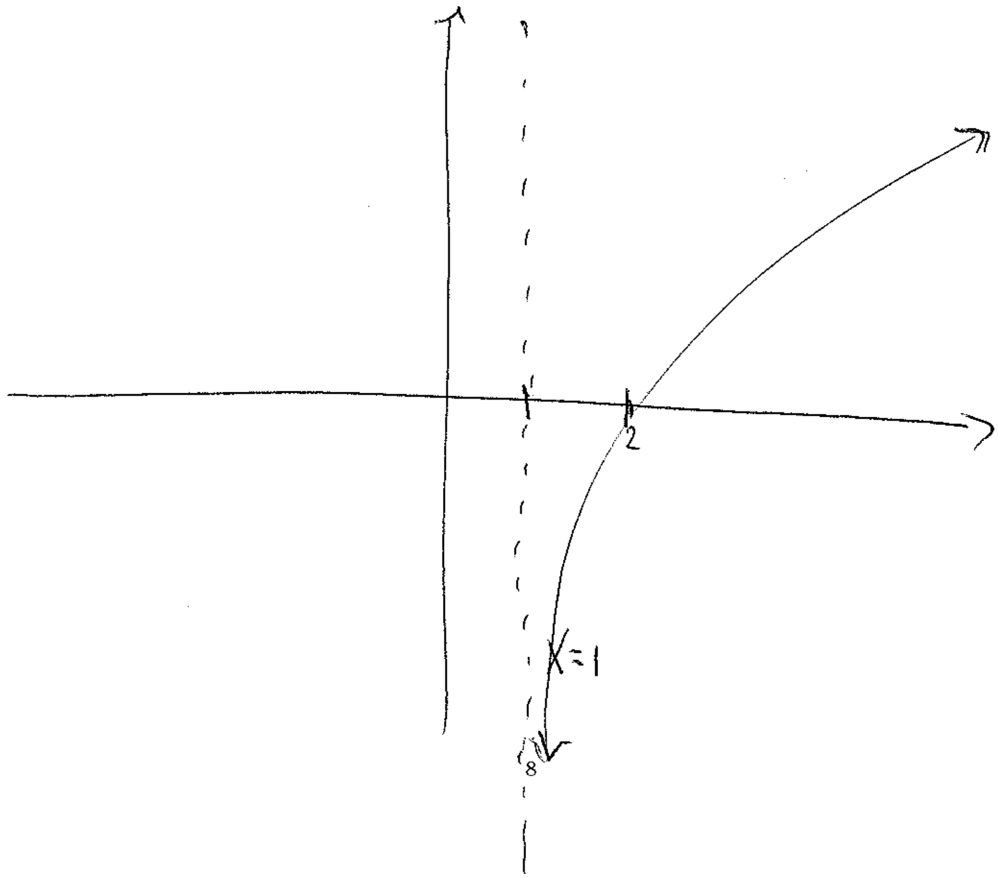


Logarithmic function, $y = \log_a x$, where $a > 0$, $a \neq 1$, and $x > 0$

Continuous at: For all $x > 0$ $\quad$ EX $\quad y = \log(x-1) \quad\quad x-1>0$

$x > 1$

Continuous on its domain, $D = (1, \infty)$



When a function is not continuous, it has one or more points where it is discontinuous:

Example. Find all values $x = a$ where the following functions are discontinuous:

1.

$$f(x) = \frac{x-1}{x^2+2x-3} = \frac{\cancel{x-1}}{(x+3)\cancel{(x-1)}} = \frac{1}{x+3}$$

Rational function

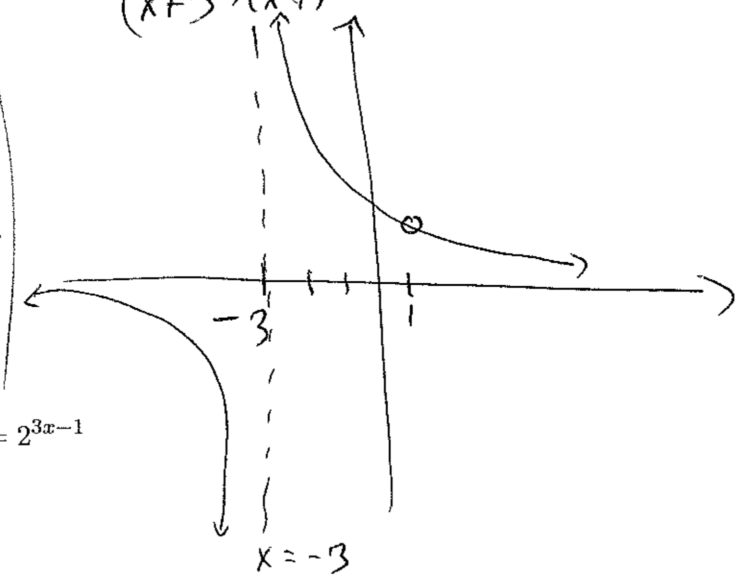
$x^2 + 2x - 3 = 0$

$(x+3)(x-1)$

$x + 3 = 0 \Rightarrow x = -3$

$x - 1 = 0 \qquad x = 1$

$f(x)$ is discontinuous at $\boxed{x = -3 \text{ and } x = 1}$



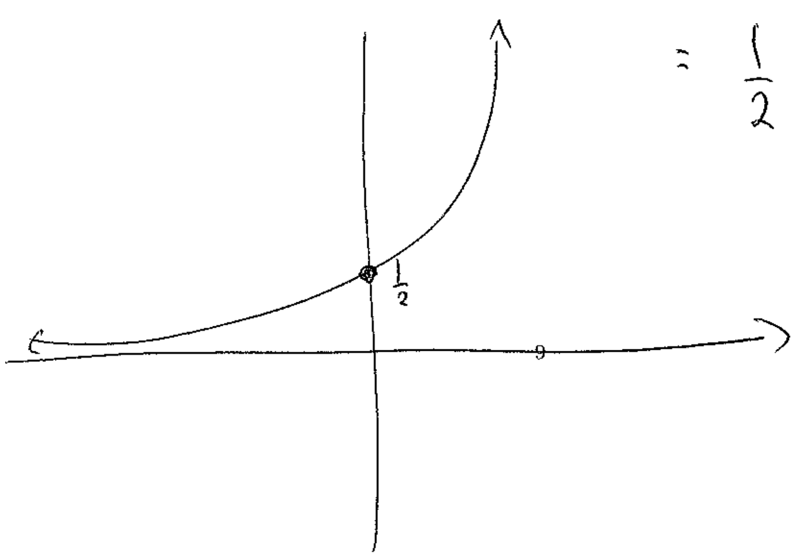
2.

$$g(x) = 2^{3x-1}$$

$g(x)$ is continuous everywhere because it is an exponential function

$$g(x) = 2^{3x-1} = 2^{-1} \cdot 2^{3x} = \frac{1}{2} \cdot (2^3)^x$$

$$= \frac{1}{2} 8^x$$



Example. Find all values of x where the piecewise function is discontinuous,

$$f(x) = \begin{cases} x+1 & \text{if } x < 1; \\ x^2+1 & \text{if } 1 \le x < 2; \\ 2x-5 & \text{if } x \ge 2. \end{cases}$$

Since each piece of this function is a polynomial, the pieces are continuous. The only X-values that f might be discontinuous are at X=1 and X=2.

Step 1. $f(1) = 2$

First consider X=1

Step 2

$$\left.\begin{aligned} \lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} x+1 = 1+1=2 \\ \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} x^2+1 = 1^2+1=2 \end{aligned}\right\} \text{Hence } \lim_{x\to 1} f(x) = 2$$

Step 3

$$f(1) = 2 = \lim_{x\to 1} f(x)$$ Thus $f(x)$ is continuous at $x=1$.

Step 2

Consider X=2.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2+1 = 2^2+1 = 5$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x-5 = 2\cdot 2-5 = -1$$

Since the left hand limit is different from the right hand limit, $\lim_{x\to 2} f(x)$ does not exists.

Hence $f(x)$ is discontinuous at $X=2$.

Thus, the point where $f(x)$ is discontinuous is at $\underline{\underline{X=2}}$ only

Example. Find the value of the constant k that makes the function continuous

$$g(x) = \begin{cases} kx^2 + 1 & \text{if } x \leq 3; \\ x + k & \text{if } x > 3. \end{cases}$$

We must have

$$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^+} g(x)$$

$$\lim_{x \to 3^-} kx^2+1 = \lim_{x \to 3^+} x+k$$

$$\lim_{x \to 3^-} kx^2+1 = k \cdot 3^2+1 = 9k+1$$

$$\lim_{x \to 3^+} x+k = 3+k$$

We must solve $9k+1 = 3+k$

$$8k = 2$$

$$k = \frac{2}{8} = \frac{1}{4}$$

Example. A car rental firm charged \$30 per day or portion per day to rent a car for a period of 1 to 4 days. Days 5 and 6 were then free, while the charge for days 7 through 10 was again \$30 per day. Let $A(t)$ represent the average cost to rent the car for t days. where $0 < t \leq 10$. Find the average cost of a rental for the following number of days:

a. 3 \$30

b. 5 $\$ \frac{4 \cdot 30}{5} = \$24$

c. 8 $\$ \frac{6 \cdot 30}{8} = \$22.5$

d. Find $\lim_{t \to 3^-} A(t)$

$= 30$

e. Find $\lim_{t \to 3^+} A(t) = 30$

f. Where is A discontinuous on the given interval?

At $t = 0, 1, 4, 5, 6, 7, 8, 9, 10$