3.2: Continuity

Continuity at $x = c$:

A function f is **continuous** at $x = c$ if the following three conditions are satisfied:

- 1. $f(c)$ is defined.
- 2. $\lim_{x \to c} f(x)$ exists, and
- 3. $\lim_{x \to c} f(x) = f(c)$.

If f is not continuous at c, it is **discontinuous** there.

We will use this 3- step test to check if a function is continuous:

Example. For the following functions, draw the graph. Then determine if the functions are continuous at the indicated x-value:

1.

$$f(x) = x + 1 \quad \text{at} \quad x = 1$$

Step 1

Step 2

Step 3

2.

$$g(x) = \frac{x^2 - 1}{x - 1} \quad \text{at} \quad x = 1$$

3.

$$h(x) = \frac{|x-2|}{x-2} \quad \text{at} \quad x = 2$$

4.

$$k(x) = \begin{cases} x+2 & \text{if } x \neq 3; \\ 4 & \text{if } x = 3. \end{cases} \qquad \text{at} \qquad x = 3$$

5.

$$l(x) = \frac{1}{x+2} \quad \text{at} \quad x = -2$$

Definition. A function is **continuous** on an open interval if it is continuous at every x-value in the interval.

Definition. A function is **continuous from the right** at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$.

Definition. A function is **continuous from the left** at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$.

Continuity on a Closed interval:

A function f is **continuous on a closed interval** $[a, b]$ if:

- 1. it is continuous on the open interval (a, b).
- 2. it is continuous from the right at $x = a$, and
- 3. it is continuous from the left at $x = b$.

Example. The function $f(x) = \sqrt{4 - x^2}$ is continuous on the closed interval $[-2, 2]$.

Here are the functions we have learned so far listed with the intervals in which the function is continuous:

Polynomial function, $y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, ..., a_1, a_0$ are real numbers:

Continuous at: For all x

Rational function, $y = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with $q(x) \neq 0$:

Continuous at: For all x where $q(x) \neq 0$

Root function, $y = \sqrt{ax + b}$, where a and b are real numbers with $a \neq 0$ and $ax + b \geq 0$:

Continuous at: For all x where $ax + b \geq 0$

Exponential function, $y = P_0 a^x$, where $a > 0$ and P_0 is the value of y at $x = 0$:

Continuous at: For all x

Logarithmic function, $y = \log_a x$, where $a > 0$, $a \neq 1$, and $x > 0$

Continuous at: For all $x > 0$

When a function is not continuous, it has one or more points where it is discontinuous:

Example. Find all values $x = a$ where the following functions are discontinuous:

1.
$$f(x) = \frac{x-1}{x^2+2x-3}$$

2.
$$g(x) = 2^{3x-1}$$

Example. Find all values of x where the piecewise function is discontinuous,

$$f(x) = \begin{cases} x+1 & \text{if } x < 1; \\ x^2+1 & \text{if } 1 \leq x < 2; \\ 2x-5 & \text{if } x \geq 2. \end{cases}$$

Example. Find the value of the constant k that makes the function continuous

$$g(x) = \begin{cases} kx^2 + 1 & \text{if } x \leq 3; \\ x + k & \text{if } x > 3. \end{cases}$$

Example. A car rental firm charged \$30 per day or portion per day to rent a car for a period of 1 to 4 days. Days 5 and 6 were then free, while the charge for days 7 through 10 was again \$30 per day. Let $A(t)$ represent the average cost to rent the car for t days. where $0 < t \leq 10$. Find the average cost of a rental for the following number of days:

a. 3 b. 5 c. 8

d. Find $\lim_{t \to 3^-} A(t)$

e. Find $\lim_{t \to 3^+} A(t)$

f. Where is A discontinuous on the given interval?