3.1: Limits

Limit of a function
Let f be a function and let a and L be real numbers. If

1. as x takes values closer and closer (but not equal) to a on both sides of a, the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to L; and

2. the value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a;
then L is the **limit** of $f(x)$ as x approaches a, written

$$\lim_{x \to a} f(x) = L.$$

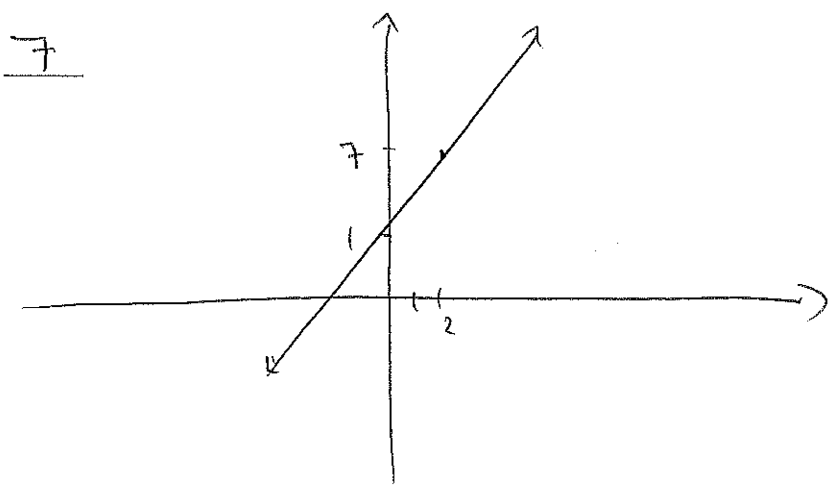
Example. Let $f(x) = 3x + 1$. We want to know what happens to $f(x)$ as x gets closer and closer to 2. We will do this by performing the following calculations:

$f(1.9) =$ 6.7 $\quad f(1.99) =$ 6.97 $\quad f(1.999) =$ 6.997

$f(2.1) =$ 7.3 $\quad f(2.01) =$ 7.03 $\quad f(2.001) =$ 7.003

We see that as x gets closer and closer to 2 from either side, $f(x)$ gets closer and closer to ____

Thus, $\lim_{x \to 2} 3x + 1 =$ 7



The **limit from the left** is written:

$$\lim_{x \to 2^-} f(x) = 7$$

The **limit from the right** is written:

$$\lim_{x \to 2^+} f(x) = 7$$

A two sided limit

$$\lim_{x\to 2} f(x) = \underline{\quad 7 \quad}$$

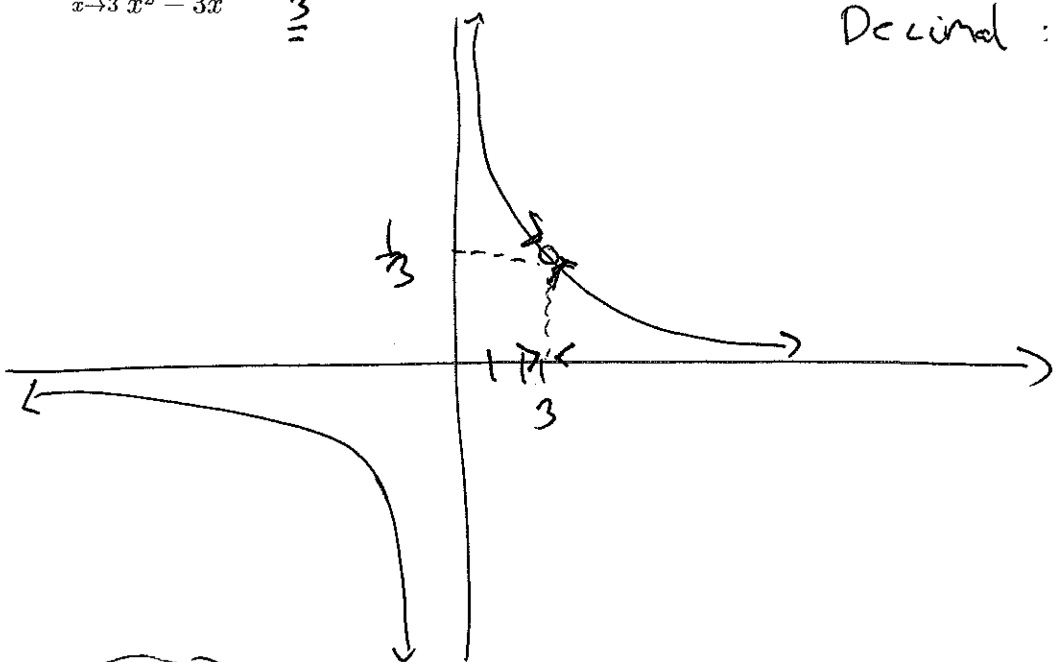
exists only if both one-sided limit exists and are the same.

Example. Find the following limits graphically:

Zoom

Decimal : 4

1. $\lim_{x\to 3} \frac{x-3}{x^2-3x} = \frac{1}{3}$



2. $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

Existence of limits
1. If $f(x)$ becomes infinitely large in magnitude (positive or negative) as x approaches the number a from either side, we write $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. In either case, the limit does not exists.
2. If $f(x)$ becomes infinitely large in magnitude (positive) as x approaches a from one side and infinitely large in magnitude (negative) as x approaches a from the other side, then $\lim_{x\to a} f(x)$ does not exist.
3. If $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = M$, and $L \neq M$, then $\lim_{x\to a} f(x)$ does not exist.

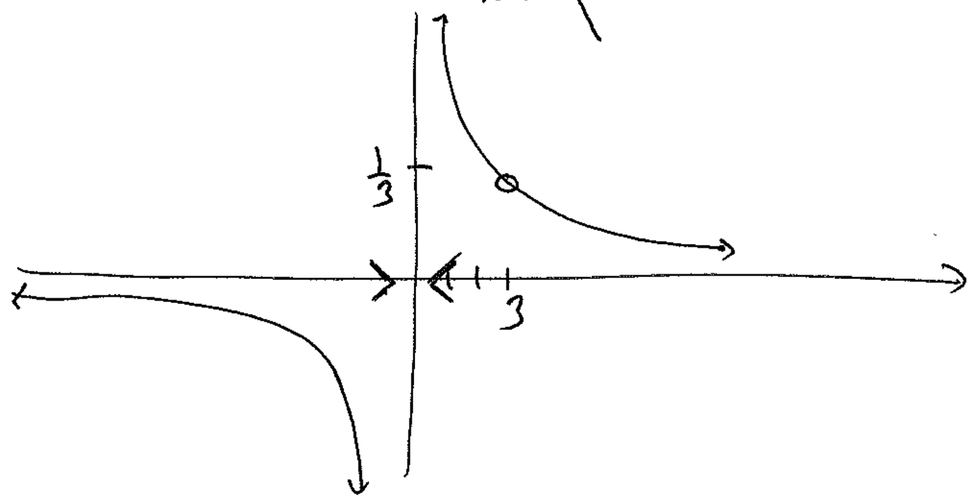
Rules for limits
Let a, A and B be real numbers, and let f and g be functions such that $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$.

- 1. If k is a constant, then $\lim_{x\to a} k = k$ and $\lim_{x\to a}[k \cdot f(x)] = k \cdot \lim_{x\to a} f(x) = k \cdot A$
- 2. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = A \pm B$
- 3. $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = A \cdot B$
- 4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{A}{B}$ if $B \neq 0$.
- 5. If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.
- 6. For any real number k, $\lim_{x\to a}[f(x)]^k = [\lim_{x\to a} f(x)]^k = A^k$, provided this limit exists.
- 7. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if $f(x) = g(x)$ for all $x \neq a$.
- 8. For any real number $b > 0$, $\lim_{x\to a} b^{f(x)} = b^{\lim_{x\to a} f(x)} = b^A$.
- 9. For any real number b such that $0 < b < 1$ or $1 < b$, $\lim_{x\to a}[\log_b f(x)] = \log_b[\lim_{x\to a} f(x)] = \log_b A$ if $A > 0$.

Example. Find $\lim_{x\to 3} \dfrac{e^x}{x-2}$ $\overset{\text{Property 4}}{=} \dfrac{\lim_{x\to 3} e^x}{\lim_{x\to 3} x-2} = \dfrac{e^3}{3-2} = \dfrac{e^3}{1} = e^3$

We evaluated this limit by just plugging in $x = 3$.

Example. Let $f(x) = \dfrac{x-3}{x^2-3x}$ $= \dfrac{\cancel{(x-3)}}{x\cancel{(x-3)}} = \dfrac{1}{x}$ for $x \neq 3$



Here there is a problem with just plugging in $x = 3$.

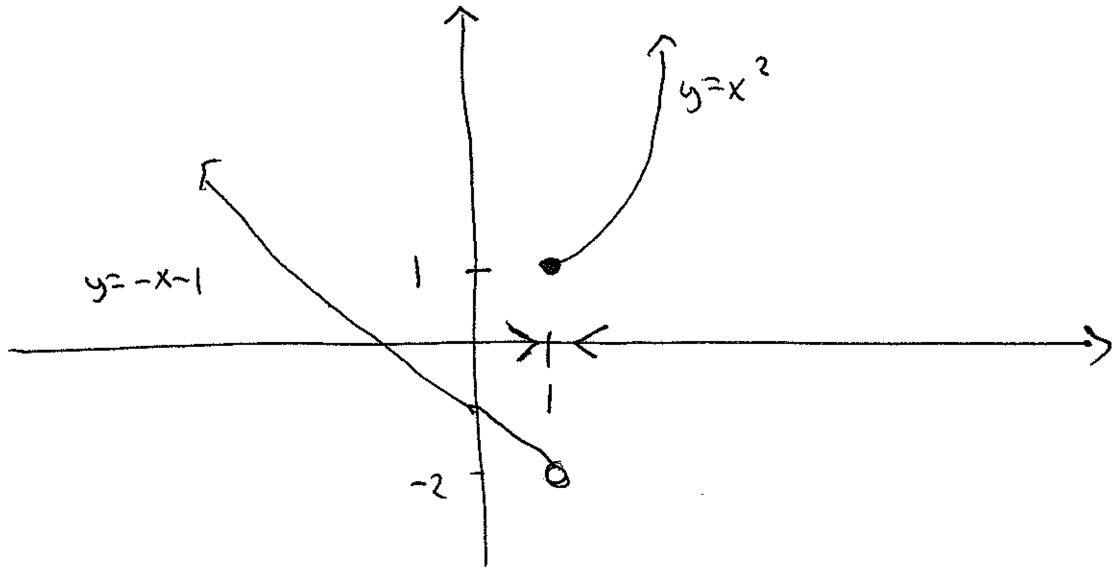
1. Find $\lim_{x\to 3} f(x)$ $= \lim_{x\to 3} \dfrac{x-3}{x^2-3x} = \lim_{x\to 3} \dfrac{1}{x} = \dfrac{1}{3}$

2. Find $\lim_{x\to 0^-} f(x)$ $= -\infty$

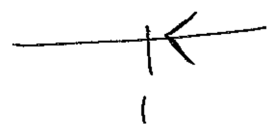
3. Find $\lim_{x\to 0^+} f(x)$ $= +\infty$

We say that $\lim_{x\to 0} f(x)$ does not exist

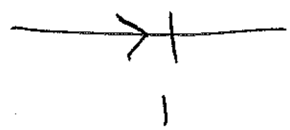
Example. Let $g(x) = \begin{cases} x^2 & \text{if } x > 1; \\ -x-1 & \text{if } x < 1. \end{cases}$



1. Find $\lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+} x^2 = 1^2 = \underline{1}$



2. Find $\lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-} -x-1 = -1-1 = \underline{-2}$



3. We have that $\lim\limits_{x\to 1} g(x)$ <u>does not</u> exists since

$$\lim_{x\to 1^+} g(x) \neq \lim_{x\to 1^-} g(x)$$

different (not equal)

Example. Find $\lim\limits_{x\to 25} \dfrac{\sqrt{x}-5}{(x-25)}$

$$= \lim_{x\to 25} \frac{(\sqrt{x}-5)}{(x-25)} \frac{(\sqrt{x}+5)}{(\sqrt{x}+5)}$$

$$= \lim_{x\to 25} \frac{\cancel{(x-25)}}{\cancel{(x-25)}(\sqrt{x}+5)}$$

$$= \lim_{x\to 25} \frac{1}{\sqrt{x}+5} = \frac{1}{\sqrt{25}+5}$$

$$= \frac{1}{5+5} = \frac{1}{10}$$

Limits at infinity:
For any positive real number n,

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Finding limits at infinity:

If $f(x) = \frac{p(x)}{q(x)}$, for polynomials $p(x)$ and $q(x)$, $q(x) \neq 0$, $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ can be found as follows:

1. Divide $p(x)$ and $q(x)$ by the highest power of x in $q(x)$.

2. Use the rules for limits, including the rules for limits at infinity,
$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$
to find the limit of the result from step 1.

Now we will return to the formal definition of a horizontal asymptote:

To find horizontal asymptotes:

The graph of the function $f(x)$ has the line $y = L$ as its **horizontal asymptote** if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

Example. 1. Let $f(x) = \dfrac{2x^2 + x - 1}{4x^2 + 3}$

Find $\lim_{x\to\infty} f(x)$

Highest power is x^2
Divide every term by x^2

$$= \lim_{x\to\infty} \frac{\cancel{\frac{2x^2}{x^2}} + \frac{x}{x^2} - \frac{1}{x^2}}{\cancel{\frac{4x^2}{x^2}} + \frac{3}{x^2}} = \lim_{x\to\infty} \frac{2 + \frac{1}{x} - \frac{1}{x^2}}{4 + \frac{3}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 2 + \frac{1}{x} - \frac{1}{x^2}}{\lim_{x\to\infty} 4 + \frac{3}{x^2}}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

$$= \frac{2 + 0 - 0}{4 + 0} = \frac{2}{4} = \frac{1}{2}$$

2. $f(x)$ has a horizontal asymptote at $y =$ $\frac{1}{2}$

3. Let $g(x) = \dfrac{2x+1}{3x^2-1}$

Divide every term by x^2

Find $\lim\limits_{x\to\infty} g(x)$

$$= \lim_{x\to\infty} \frac{\frac{2x}{x^2} + \frac{1}{x^2}}{\frac{3\cancel{x^2}}{\cancel{x^2}} - \frac{1}{x^2}} = \lim_{x\to\infty} \left(\frac{\frac{2}{x} + \frac{1}{x^2}}{3 - \frac{1}{x^2}} \right)$$

$$= \frac{0+0}{3-0} = \frac{0}{3} = \underline{0}$$

4. $g(x)$ has a horizontal asymptote at $y =$ __0__