3.1: Limits

Limit of a function
Let f be a function and let a and L be real numbers. If

1. as x takes values closer and closer (but not equal) to a on both sides of a, the corresponding values of $f(x)$ get closer and closer (and perhaps equal) to L; and

2. the value of $f(x)$ can be made as close to L as desired by taking values of x close enough to a;

then L is the **limit** of $f(x)$ as x approaches a, written

$$\lim_{x \to a} f(x) = L.$$

Example. Let $f(x) = 3x + 1$. We want to know what happens to $f(x)$ as x gets closer and closer to 2. We will do this by performing the following calculations:

$f(1.9) =$ $\qquad$ $f(1.99) =$ $\qquad$ $f(1.999) =$

$f(2.1) =$ $\qquad$ $f(2.01) =$ $\qquad$ $f(2.001) =$

We see that as x gets closer and closer to 2 from either side, $f(x)$ gets closer and closer to ______

Thus, $\lim_{x \to 2} 3x + 1 =$ ______

The **limit from the left** is written:

$$\lim_{x \to 2^-} f(x) = \underline{\qquad}$$

The **limit from the right** is written:

$$\lim_{x \to 2^+} f(x) = \underline{\qquad}$$

A **two sided limit**

$$\lim_{x \to 2} f(x) = \_\_\_\_\_\_$$

exists only if both one-sided limit exists and are the same.

Example. Find the following limits graphically:

1. $\displaystyle\lim_{x \to 3} \frac{x-3}{x^2-3x}$

2. $\displaystyle\lim_{x \to 0} (1+x)^{\frac{1}{x}}$

Existence of limits
1. If $f(x)$ becomes infinitely large in magnitude (positive or negative) as x approaches the number a from either side, we write $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$. In either case, the limit does not exists.
2. If $f(x)$ becomes infinitely large in magnitude (positive) as x approaches a from one side and infinitely large in magnitude (negative) as x approaches a from the other side, then $\lim_{x\to a} f(x)$ does not exist.
3. If $\lim_{x\to a^-} = L$ and $\lim_{x\to a^+} = M$, and $L \neq M$, then $\lim_{x\to a} f(x)$ does not exist.

Rules for limits
Let a, A and B be real numbers, and let f and g be functions such that $\lim\limits_{x\to a} f(x) = A$ and $\lim\limits_{x\to a} g(x) = B$.

- 1. If k is a constant, then $\lim\limits_{x\to a} k = k$ and $\lim\limits_{x\to a}[k \cdot f(x)] = k \cdot \lim\limits_{x\to a} f(x) = k \cdot A$
- 2. $\lim\limits_{x\to a}[f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = A \pm B$
- 3. $\lim\limits_{x\to a}[f(x) \cdot g(x)] = \lim\limits_{x\to a} f(x) \cdot \lim\limits_{x\to a} g(x) = A \cdot B$
- 4. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{A}{B}$ if $B \neq 0$.
- 5. If $p(x)$ is a polynomial, then $\lim\limits_{x\to a} p(x) = p(a)$.
- 6. For any real number k, $\lim\limits_{x\to a}[f(x)]^k = [\lim\limits_{x\to a} f(x)]^k = A^k$, provided this limit exists.
- 7. $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$ if $f(x) = g(x)$ for all $x \neq a$.
- 8. For any real number $b > 0$, $\lim\limits_{x\to a} b^{f(x)} = b^{\lim_{x\to a} f(x)} = b^A$.
- 9. For any real number b such that $0 < b < 1$ or $1 < b$,
  $\lim\limits_{x\to a}[\log_b f(x)] = \log_b[\lim\limits_{x\to a} f(x)] = \log_b A$ if $A > 0$.

Example. Find $\lim\limits_{x\to 3} \dfrac{e^x}{x-2}$

We evaluated this limit by just plugging in $x = 3$.

Example. Let $f(x) = \dfrac{x-3}{x^2-3x}$

Here there is a problem with just plugging in $x = 3$.

1. Find $\lim_{x \to 3} f(x)$

2. Find $\lim_{x \to 0^-} f(x)$

3. Find $\lim_{x \to 0^+} f(x)$

We say that $\lim_{x \to 0} f(x)$____________

Example. Let $g(x) = \begin{cases} x^2 & \text{if } x \geq 1; \\ -x-1 & \text{if } x < 1. \end{cases}$

1. Find $\lim_{x \to 1^+} g(x)$

2. Find $\lim_{x \to 1^-} g(x)$

3. We have that $\lim_{x \to 1} g(x)$___________

Example. Find $\lim_{x \to 25} \dfrac{\sqrt{x} - 5}{(x - 25)}$

Limits at infinity:
For any positive real number n,

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{1}{x^n} = 0$$

Finding limits at infinity:

If $f(x) = \frac{p(x)}{q(x)}$, for polynomials $p(x)$ and $q(x)$, $q(x) \neq 0$, $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$ can be found as follows:

1. Divide $p(x)$ and $q(x)$ by the highest power of x in $q(x)$.

2. Use the rules for limits, including the rules for limits at infinity,

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{1}{x^n} = 0$$

to find the limit of the result from step 1.

Now we will return to the formal definition of a horizontal asymptote:

To find horizontal asymptotes:

The graph of the function $f(x)$ has the line $y = L$ as its **horizontal asymptote** if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

Example. 1. Let $f(x) = \dfrac{2x^2 + x - 1}{4x^2 + 3}$

Find $\lim_{x\to\infty} f(x)$

2. $f(x)$ has a horizontal asymptote at $y =$ ______

3. Let $g(x) = \dfrac{2x+1}{3x^2-1}$

Find $\lim_{x \to \infty} g(x)$

4. $g(x)$ has a horizontal asymptote at $y =$ ______